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Eight Quarterly Progress Report

Covering the Period:

6/1/61 - 8/31/61

Title

ANALYTICAL STUDY OF THE BEHAVIOUR OF FLUIDS IN POROUS SOLID MEDIA

ASTIA

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Prepared by:

Walter Rose
Project Director

December 5, 1961

Abstract Card

AD_____Accession No. _____   1. Capillarity
2. Porous media
hydrodynamics

The Board of Trustees of the
University of Illinois
ANALYTICAL STUDY OF THE BEHAVIOUR
OF FLUIDS IN POROUS SOLID MEDIA

Walter Rose

Report No. 8, 5 December, 1961 pp __113__
Contract No. DA18-108-405-CML-517

Progress is reported for work undertaken in the following
areas: (1) Synthesis of the dynamics of capillary imbibition,
(2) Discussion of the form of fluid-fluid interfaces in un-
steady motion, (3) Discussion of the problem of viscous
fingering, (4) A statement of the physics of immiscible
fluid-fluid displacement, (5) Network model theory, (6) Generalized
flow problems, (7) Elementary cases of capillary flow,
(8) Spectural analysis of the structure of porous media,
(9) Numerical integration of the equations of Stokes-Navier.

EIGHT QUARTERLY PROGRESS REPORT

TABLE OF CONTENTS

Preface
Summary of Progress
Supplementary Statements

PREFACE

Shortly after the seventh progress report was completed, the principal investigator (while retaining supervisory control) withdrew his active participation in the Contract work because, even with a supplemental allowance received in July, there was not enough money to cover more than the summer salaries of the permanent project workers (research associates and assistants). In due course it was subsequently learned that the support of the work would terminate August 31, and the task to prepare a final report followed as a consequence.

With the return of the principle investigator to the Illinois campus in September, it was considered premature to prepare the required final report. For example, the Masters thesis work of N. Chaudhari was to be written in the fall as based on the experimental data obtained during the summer. H. Para, who had a half-time appointment from February through August, was still in the process of crystallizing his ideas about network model studies, integration of the Stokes-Navier equations, and about the meaning of Brittin's work. The situation with W. Rose was that he had a number of partly-finished manuscripts which needed additional work before they could be considered for publication. In fact, the only project worker in a position to write a summary statement was R. Channapragada (who was committed in advance to terminate at the end of the summer, and who was committed therefore to leave a statement of his progress over the two-year period of his contract work). Unfortunately, Dr. Channapragada did not find the time to prepare the wanted definitive summary of his work.

The form of the final progress report as attached reflects on the factors mentioned above. A completely rewritten Fara and Rose paper is included, together with a statement by Fara about his ideas on programming solutions of the Stokes-Navier equations, solutions of certain categories of network model studies, and other questions. An abstract of the thesis of N. Chaudhari is included. A number of interim statements are offered by W. Rose, and finally there appears the brief statements of Dr. Channapragada.

In due course, supplemental statements will be offered to complete the material as offered in this final report. That is, we take the position that our continuing work under new sponsorship had its origin in the CML contract work, and that an obligation rests on us to provide complete statements at the end of all projects started under the CML sponsorship. Specifically, the final version of Chaudhari's thesis will be offered, together with revisions of certain of the manuscripts of W. Rose and others contained herein.

SUMMARY OF PROGRESS

Over the two-year period, September 1959 through
August 1961, a project titled "Analytic Study of the Behavior
of Fluids in Porous Media" was supported by the Chemical
Warfare Laboratories (Directorate of Research) at the
University of Illinois. This work is identified by Contract
Number DA-18-108-405-CML-517, and has led to the preparation
of eight quarterly progress reports of which this is the last
and final report.

During the subject period, some 20 students, post-docto-
rates, consultants and faculty were employed for varying
periods of time, cumulatively resulting in the support of
some seven man-years of effort. Three masters thesis were
completed under contract sponsorship, and some one dozen
manuscripts were prepared and are being prepared and approved
for publication.

It is clear from the foregoing that the contract funds
were used largely for support of students, of which six were
doctorate candidates. The principal research associate,
R. Channapragada, for example, received his PhD degree
(in mathematics) in June 1961. Less than two percent of the
contract funds has been used for the purchase of equiptment
and laboratory supplies.

The contract work, while it has led to some new scientific
findings (all of which are either now published or soon to be
published), has been particularly meaningful in forming a
foundation for future activities and research studies to be
undertaken in this Laboratory forthwith under National
Science Foundation and other Department of Defense support.
We list the major scientific achievements of the contract
work, as follows:

1. The development of a better understanding of the interconnection between the micro and macroscopic aspects of the unsteady-states of mixture flow in porous media. (cf. Rose and Channapragada, III and IV*; Rose VIII);

2. An investigation of the role of electrokinetics in fluid flow. (cf. Street, V; Street and Rao, IV; Stewart and Street, IV; Street and Stewart, IV; Street, III, etc.).

3. An experimental investigation of the velocity and acceleration dependency of moving contact angles. (Chaudhari, VIII; Rose, VII; Heins, VII; Chaudhari, VII; Rose and Chaudhari, VI; Rose and Heins, V; Rose, Ramagopal and Heins, IV; Heins, IV, etc.).

4. An investigation of the meaning and use of the Brittin equation to describe capillary flow. (cf. Fara and Rose, VIII; Fara and Rose, VII; Tung and Drews, VI; Rose and Fara, VI; Fara, Rose and Tung, V; Tung, IV; etc.).

5. A consideration of the solution of network model problems. (cf. Fara, VIII; Fara, VII; Tung and Rose, V; Fara and Rose, V: Rose, IV; etc.).

As for the continuing studies in progress in this laboratory which represent an outgrowth of the CML-517 work, we refer particularly to the statement of Rose (VII) titled: "The Basic Problem Reformulated". Here it is made clear that to even begin to meet the broad objectives contained in the original preamble to the CML-517 project, an appeal must be made to the fundamental fluid mechanics and surface chemistry of porous media systems. Indeed, it was in the spirit of this view to which the various general contributions of Channapragada were directed (cf. contributions in VIII, VI, V and IV in particular).

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*Refers to the contribution of Rose and Channapragada as it appeared in the third and fourth quarterly progress reports, etc.

We conclude that the problems of highest priority to which we should now direct our attention, include:

    1. Numerical means to accomplish a "brute-force" integration of the equations of Navier-Stokes, with major nonlinearities retained (cf. Fara and Geiger, VIII).

    2. A further study of how charged double layers (i.e. the electrokinetic effects) control the shapes of interfaces in motion, as compared to the influence of the hydrodynamic effects.

    3. The development of a general theory of immiscible fluid displacement in porous media which accounts for the observation of viscous fingering, (cf. Rose, VIII).

ACKNOWLEDGMENT

SYNTHESIS OF THE DYNAMICS OF CAPILLARY IMBIBITION
By
Walter Rose

1). Given, a single (peripherally closed) pore channel of specified volume, V, defined by the equation:

$$S = S \ (x,y,z)$$

Thus, the space coordinates of all points lying on the surface of the pore channel are known; moreover, the pore channel will have entry and exit "ends" as defined by the closed space curves which superimpose as terminal positions on the surface S. (See figure 1);

2). As an initial condition, let us assume that the pore channel contains some volume of wetting fluid, say V_w which is smaller than V, and is elsewhere filled with a nonwetting fluid (e.g. the saturated vapor of the wetting liquid), so that it can be said (at all subsequent times as well as initially):

$$V = V_w + V_n$$

To further specify the initial condition, it is assumed that both the entry and exit ends of the channel are temporarily closed by immaginary barriers, and that a sufficiently long period of time has elasped so that the fluids have assumed a spatial configuration which makes the free surface energy of the system a minimum value.

The first problem, therefore, is to specify this minimum energy configuration by finding the equation of the fluid-fluid interfacial surface. This solution in the limit will also give the closed space curve which defines

the contact of the fluid-fluid interface with the surface, S.

The free surface energy of the pore channel system with its contained fluids (for all configurations) is:

$$E = \sum_i \gamma_i \sigma_i , \quad [\, i=sw, sn, wn \,]$$

where the γ's are the interfacial tensions (ergs per unit area), and the σ's are the surface areas of the two solid-fluid and the one fluid-fluid interfaces. A second useful relationship is Young's equation, namely:

$$\cos \theta_e = (\gamma_{sn} - \gamma_{sw}) / \gamma_{wn}$$

which says that at equilibrium, the angle of contact, θ_e, made by the fluid-fluid interface with the surrounding pore channel wall will have a fixed and invariant value. Now, since the function, S, is known, the total surface area of the pore channel, σ_T, can be calculated. Also to be noted is the fact that

$$\sigma_T = \sigma_{sw} + \sigma_{sn}$$

Thus, the configuration of minimum surface energy is found by applying the methods of the calculus of variations, setting $\Delta E = 0$, which reduces to:

$$\gamma_{sn} \sigma_{sn} + \gamma_{sw} \sigma_{sw} + \gamma_{wn} \sigma_{wn} = 0$$

$$- (\gamma_{sn} - \gamma_{sw}) \Delta \sigma_{sw} + \gamma_{wn} \Delta \sigma_{wn} = 0$$

$$- (\gamma_{wn} \cos \theta_e) \Delta \sigma_{sw} + \gamma_{wn} \Delta \sigma_{wn} = 0$$

or, $$\Delta \sigma_{wn} = \cos \theta_e \, \Delta \sigma_{sw}$$

In the calculation being considered, gravity (and other body forces) are assumed to be absent; therefore, a further simplifying condition can be imposed, namely, that the

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*sw designates the solid-wetting fluid interface, sn designates the solid-nonwetting fluid interface, wn designates the wetting-nonwetting fluid interface.

equilibrium configuration of the fluid-fluid interface
will be a constant-curvature surface, or:

$$(1/r_1) + (1/r_2) = K$$

where K is the constant curvature of the interface at
every point and the r's are the principal radii of
curvature at every point.

In general, it may be expected that if no path of
approach is postulated, there will be one (and only one)
configuration of minimum surface energy which exists
consistant with the geometry of the pore channel (as
defined by the function, S, and the parameter V), with
the initial value of V_w, and with the constants of the
system (i.e. the γ's and the value of θ_e). That is,
no matter how the wetting liquid (as measured by the
volume, V_w) is introduced, paths <u>via</u> diffusion* exist
when the time-period is long which make it possible for
a truly minimal energy condition to be achieved. (Note,
if the approach to equilibrium were by a flow path alone,
one could expect in the general case that energy "barriers"

---

* and capillary condensation (adsorption).

would be encountered, which is to say that a metastable minimum energy configuration would be attained as related to the approach path)[*].

The final step at this stage of the calculation is to derive the curvature of the fluid-fluid interface from a determination of the equation of the wn-interface, from which is then deduced the magnitude of the pressure discontinuity associated with the interfacial curvature by the Laplacian relationship:

$$\Delta P = \gamma_{wn} K.$$

3). We next assume that an infinitesimal additional amount of wetting fluid is introduced, as measured by ΔV_w, so that the new volume of wetting fluid is $(V_w + \Delta V_w)$. For the purposes of conceptual visualization, it will be desirable to have limited in the first place consideration to that limited class of pore channel geometries characterized

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[*] The analogy of a ball "rolling down a pocketed hill" will clarify this point. The kinetic energy of the ball is disregarded, and the potential energy of the ball is taken as c corresponding to the surface energy of the capillary channel system. In this case, the ball rolls to the first pocket (energy barrier) and stops, as long as its approach to "equilibrium" is limited by saying that the approach path of the ball must be maintained on the surface of the hill. To bypass the trap, external work must be done; otherwise, an independent, secondary path of approach to a lower equilibrium must be postulated. In the capillary tube system, vapor diffusion and capillary condensation afford this independent path. That is, the interface can spontaneously move forward by flow as long as the interface energy of the system is always decreasing. The flow then stops (say, temporarily) whenever a further forward movement means that the energy of the system is increased (i.e. the ball is in the bottom of a pocket); however, if sufficient time is given, the wetting fluid can be transportated ahead via diffusion and capillary condensation, to provide for a complete approach to a state of minimum surface energy. The forward wetting fluid may or may not later coalesce with the main body of wetting fluid.

so that the equilibrium configuration of the initial
volume, V_w, of the wetting fluid is located (at least in
part) continuous with the so-called entry end of the
channel.

In any case, we now seek to find the new (minimum sur-
face energy) configuration of the new volume of wetting
fluid (i.e. $V_w + \Delta V_w$), by the methods outlined in
paragraph two above. In this case, however, it will be
desirable to limit first attention to the configuration
for which an approach path (in the flow sense) is obviously
extant; however, immediately a great difficulty is encountered.
For example, even though ΔV_w is made very small
(approaching zero), the very act of this addition
necessarily pushes the fluid-fluid interface forward,
after which a minimum surface energy configuration is to
be sought. However, in the general case, the geometry
of the pore space (as identified by the function, S)
just ahead of the initial position of the fluid-fluid
interface may be such that external work will have to be
done (to an underminable extent) to exceed the threshhold
value of some incipient energy barrier.
The foregoing is a problem since it is the ultimate
objective of this work to describe the spontaneous
capillary imbibition of a wetting fluid into a pore
channel of given geometry. This is to be done by
noting the curvature of the fluid-fluid interface
after each incremental addition of a volume of wetting
fluid, V_w, from which can be derived the time-changing
value of grad P. The latter is then utilized as the
driving force term appearing in the Navier-Stokes equation,
which upon integration gives the rate-of-interface movement.

In capillary flows of the type being considered, gravity
(and other body forces), and pressure forces may be
acting together with the surface forces to produce motions
of interest. It seems necessary however, to first deduce
the magnitude and influence of the capillary (surface)
driving force as acting alone as these relate in a complex
way to the function, S. Once they are determined, then
the added effect of supplying pressure energy and of
considering the action of external body forces, :an be
accounted for by simple superposition.

It seems desirable, therefore, to limit attention first
to a pore energy geometry such that the function, S,
will define a surface whose cross-sectional perimeter
will increase continuously when proceeding from the entry
end (where initially the wetting fluid will be located)
to the exit end. Indeed, because of the above considerations,
and to achieve additional simplicity, we may say that it
will be desirable to first limit attention to a capillary
channel which is symmetrical around, say, the z-axis,
and whose radial coordinate, say R, increases continuously
from z=0 to z=L (where L is the length of the channel).
With the above simplification, the surface S will be
characterized by the function, $R = R(z)$, so that when
dR/dz is small, large values of ΔV_w can be considered,
and when dR/dz is large, the values of ΔV_w to be con-
sidered will be correspondingly reduced to gain in
resolution of the time-changing magnitude of the capillary
driving force.

exit end
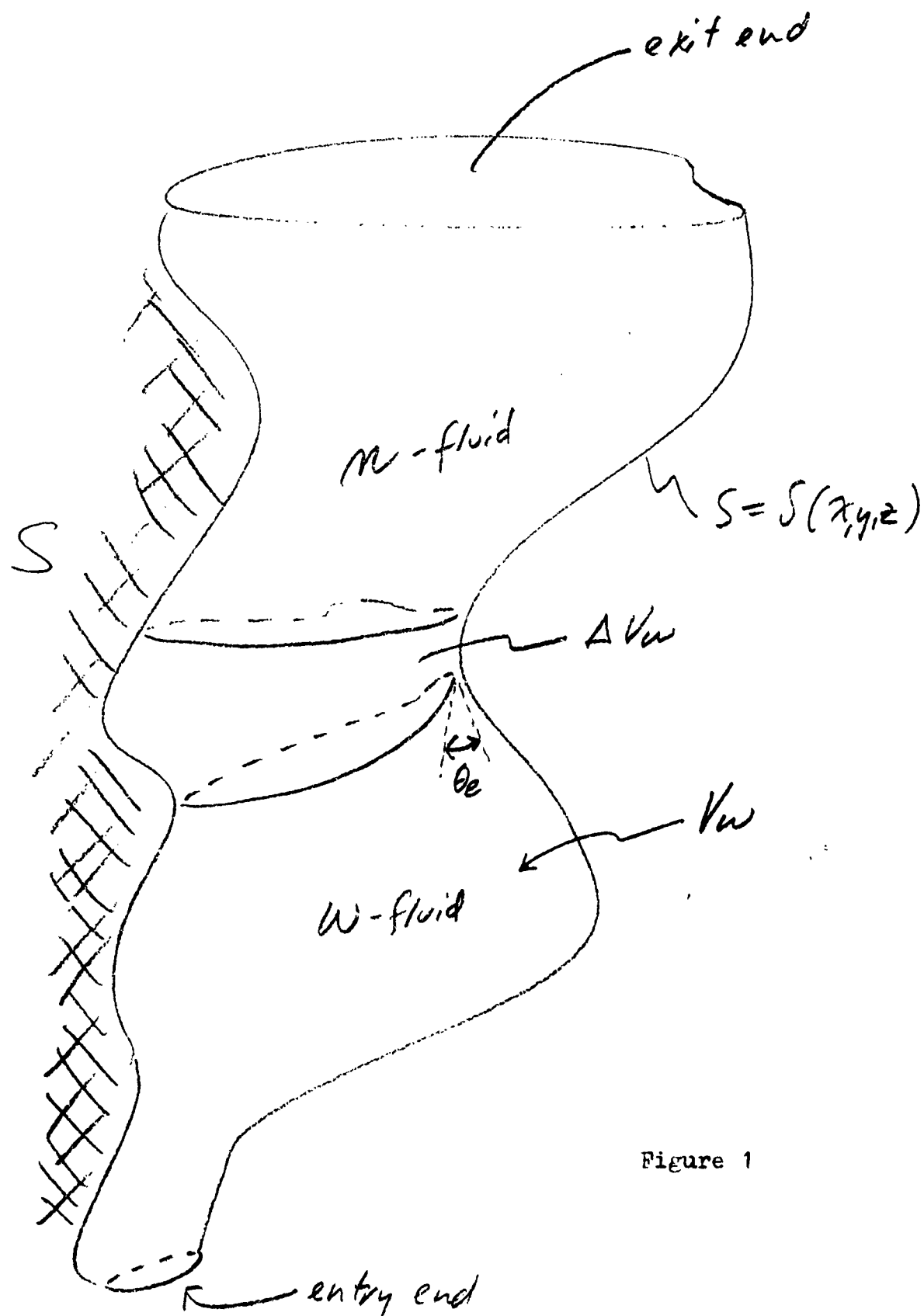
n-fluid

S

$S = S(x, y, z)$

ΔV_w

θ_e

V_w

W-fluid

Figure 1

entry end

FLUID-FLUID INTERFACES IN UNSTEADY MOTION
By
Walter Rose, Harry Para and N. Chaudhari

In a recent paper[1] reference was made to how a constant curvature interface could be expected as defining the junction of two immiscible fluids in steady motion through a capillary tube of constant cross-section. With this assumption, Rose and Heins[2] took the contact angle, θ_A, as a measure of the curvature of advancing interfaces, and they reported data for the steady flow of certain liquids in circular capillaries which seemed to fit the linear relationship:

$$\cos \theta_A = \cos \theta_E + M_v Z' \qquad (1)$$

where the independent variable, Z', is the velocity of interface movement, the slope, M_v, characteristically is negative, and the intercept, $\cos \theta_E$, is the cosine of the equilibrium contact angle as defined by Young's equation. We now prepare to test Equation (1) against the data of LeGrand and Rense[3].

The contact angle rate-dependency given by Equation (1) has a form consistant with the early conclusions of Ablett[4] and the later speculations of Barrer[5]; indeed, Siegel[6] independently has made use of the equivalent of Equation (1) to explain some of his findings. Since values of cosine θ_A less than -1 are not allowed, however Equation (1) in the most favorable cases will only be descriptive of observed data at low values of Z'

It is our purpose now to show that, as a useful approximation, Equation (1) can be expanded in the form of a Maclaurin Series when acceleration as well as velocity dependency is to be taken into account. Thus:

$$\cos \theta_A = \cos \theta_E + M_v Z' + M_a Z'' +$$

$$(\text{½}) \left[M_3 (Z')^2 + M_4 (Z'')^2 + M_5 Z' Z'' \right] + //// \qquad (2)$$

where the independent variable, Z'', is the acceleration of the interface movement, and the M-co-efficients are the time-zero limits of the partial derivatives of $\cos \theta_A$ with respect to the associated independent variables, Z' and Z''.

Let us consider vertical rise an incompressible of wetting liquid in circular capillaries in the manner of Brittin[7] (that is, neglecting the viscosity and inertia of the vapor ahead of the imbibing liquid, neglecting the entry turbulence, employing the Poiseuille steady-state viscous resistance term in the unsteady-state system, etc.). We may then write:

$$(Z Z')' + A (Z Z') + B Z + C \cos \theta_A = 0 \qquad (3)$$

where $\quad A = 8 \mu / \rho R^2$

$\qquad B = g$ (i.e. the acceleration due to gravity)

$\qquad C = -(2 T / \rho R)$

and where μ is the viscosity, ρ is the density, and T is the surface tension.

Taking only the first three terms of Equation (2) as an approximation and combining with Equation (3), gives an integration:

$$(Z Z') + (A/2) (Z^2) + B \int z \, dt = C \left[\cos \theta_E t + M_v Z + M_a Z' \right] \qquad (4)$$

With the aid of Equation 4, then, the constants, M_v and M_a, can be evaluated from a set of experimental data of Z as a function of time, t, by a least-squares analysis involving the simultaneous solution of the following set of equations:

$$0 = \sum (A_i + M_v + B_i M_a)$$

$$0 = \sum (A_i + M_v + B_i M_a)(B_i)$$

<div style="text-align:right">(5)</div>

where A_i is the value of the left-hand member of Equation (4) at time t_i, and B_i is the ratio of Z' to Z at time t_i.

In Table I we give the values of M_v and M_a which have been calculated by Equations (5) from the data obtained in the nine capillary rise experiments reported by LeGrand and Rense[3]. The following points are of interest:

1). Values of M_a characteristically are one-tenth the corresponding values of M_v; therefore, one apparently is justified in dropping the fourth and higher terms of Equation (2).

2). In a decelerating system, the contact angle is greater at a particular velocity than if the flow were a steady flow at the same velocity, therefore, M_v is characteristically negative and M_a characteristically is positive*.

3). During the initial impulsive motion as Z' goes from zero to some maximum value, Z'' is positive, so that the term, $M_a Z''$ serves as a check on the term, $M_v Z'$, and unreasonable values of cosine θ_A are not predicted by Equation (2). N.B. Compensation also afforded by other neglected terms.

4). M_v seems to correlate with the dimensionless group formed as the ratio of the viscous to capillary forces. That is, the greater the viscosity (other things being equal), the greater will be M_v; and, conversely, the greater the surface tension (other things being equal) the smaller will be M_v.

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* Indeed, it is as if the decelerating interface "remembers" the higher previous value of contact angle.

This note on unsteady motions, like the previous one on steady motions[1], in a heuristic way, gives information about the curvature of fluid-fluid interfaces moving in capillary channels.

TABLE I +

LeGrand and Rense
Experiment Number

3 (water in 0.35 cm. tube) $M_w = 0.037$ $M_a = 0.0020$

4 (EtOH in 0.0242 cm. tube) $M_w = 0.028$ $M_a = 0.0018$

+ The complete table is not presented at this time inasmuch as the values obtained by desk calculator operating on graphically determined values of Z' and on planimeter-values of the integral of Zdt, were not showing the desired resolution. Inddue time, a machine programmed solution of Equation 5 will be available. To be noted, none the less, is the fact that the values given above correspond closely to those given in Tables I and II of the 13th Supplementary Statement of this Progress Report.

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VISCOUS FINGERING AS A FEATURE
OF STABLE DISPLACEMENT
By
Walter Rose

1. Historical Considerations
2. Microscopic Displacement Theory
3. Problems in Macroscopic Representation
4. Qausi-instability analogies

Abstract

In recent years it has become popular to suppose that the stability of a displacement front is determined by the magnitude of the so-called mobility ratio and other associated factors. While agreement has not been obtained on all points, the summaries presented by Scheidegger (1960) and Collins (1961) may be taken as the representative views which current book authors choose to advance.

In this Note, we take exception to these popular theories which account for the observation of viscous fingering as a manifestation of instability phenomena. First, we emphasize that it is a viscosity ratio condition, and not a mobility ratio condition, which must be met before viscous fingering will be observed. Second, we emphasize that the flood front is not a well-defined surface* to which instability considerations are to be applied, but rather it is simply a moving-zone region having a high density of microscopically distributed fluid-fluid interfaces, which region is marked by a high normal saturation gradient. Third, we emphasize that the physico-chemical interactions and the laws of motion which apply to the description of displacement systems, at least in the microscopic sense, are no different for high viscosity ratio (and/or mobility ratio) systems than for low viscosity ratio (and/or mobility ratio) systems, therefore, the analysis

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* For example, analogous to the Hele-Shaw cell interface discussed by Saffman and Taylor (1958).

and description of viscous fingering entails no special
difficulties beyond those met in ordinary (so-called stable)
displacement problems.

As for the first point, experiment shows that viscous
fingering is never observed unless the viscosity of the entering
fluid is less than that of the replaced fluid, no matter what
value the mobility ratio may have had during the experiment.
As for the second point, it can be shown that the wave length
of incipient fingers corresponds to the wave length of micro-
heterogeniety, and both involve linear dimensions comparable
to those which measure the areial extent of the various
fluid-fluid interfaces in motion (i.e. a few pore diameters
at the most between contiguous pore wall contacts); therefore,
once can talk only about a condition of quasi-instability in
analogy, for example, to the unstable configuration of the
interface seen in the Hele-Shaw cell, and in miscible fluid
porous media displacement systems when the entering fluid is
less viscous than the replaced fluid. Finally, as for the
third point above, we show herein that the Buckley-Leverett
procedure for describing the unsteady states of immiscible
fluid mixture flow, can be applied to represent viscous fingering
with the same degree of success as experience shows it applies
to the so-called stable displacement processes.

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Physics of Immiscible Fluid-Fluid Displacement
By
Walter Rose

1.   Introduction

The physics of flow through porous media is a subject
of recognized importance and wide popularity. It is also a
subject of great complexity, and even the recent book authors
(Polubarinova-Kochina, 1952; Carman, 1956; Houpeurt, 1957;
Lykow, 1958; Scheidegger, 1960; Collins, 1961) acknowledge
that there is still much to be learned. In any case,
view is adopted here.

To visualize the problem fully, one must become acquainted
with the general features of transport phenomena as discussed
for example by Bird et al (1960), and with the particulars of
theoretical fluid mechanics as treated for example by Landau
(1959). With this foundation, one can begin to construct the
equations of motion whose integrals are wanted to predict the
details of porous media flows.

It is the purpose of this paper to show that even when we
limit attention to rather simple and elementary cases, great
difficulties in analysis often are encountered. These may
occur because the porous media geometries, in general, are not
specifiable; moreover, the large characteristic surface area
of the pore structure gives rise to unusual viscous drag effects
and to unique capillary (fluid-fluid-solid) interactions. In
consequence, there is a great temptation and tendency to over-
simplify the mathematical model chosen to represent the porous
media flow system, leading to the categories of uncertainty
already discussed with great insight by Massé (1958).

Let us limit attention arbitrarily to some simple dis-

placement problem where one fluid is seen to replace another
immiscible fluid from the pore space of an interconnected net-
work of capillary channels (i.e. a porous medium). The widely
accepted solution to this problem is that given in $\S$2 of this
paper, which solution, however, is believed to be suspect on
three counts, namely:

1.   No explicit account is taken in the general theory
as referring to the onset of instabilities believed by
some to develop at the later stages of displacement (i.e.
when the mobility of the replacing fluid finally becomes
higher than that of the replaced fluid). This matter is
analyzed in $\S$3 of this paper.

2.   The general theory is derived from the untested
assumption that the unsteady-states of multiphase flow
can be described by reference to a Darcy law analog. In
$\S$4 this assumption is analyzed and then abandoned.

3.   The general theory treats the action of capillary
forces in a way which is unnecessarily inexact, as dis-
cussed in $\S$5 of this paper.

The three topics, enumerated above, are introduced here
heuristically inasmuch as they have been largely ignored by
previous authors. For example, there is no mention of them in
the books cited above. While it must be admitted that the
general theory which ignores these points gives good results
in some cases, it will be shown that modifications must be in-
troduced to predict correctly the course of other practical
cases of porous media flow.

1.   No explicit account is taken in the general theory which
will represent the onset and growth of viscous fingering which
is seen to occur when certain conditions are met (including
the condition that the replacing fluid is more mobile than the
displaced fluid. It is true that quasi-instability considerations
have been introduced by some, but in § 3 of this paper it is
shown how the latter approach begs rather than resolves the
question.

2.   Statement of the Problem

Suppose we have a porous medium with given boundary
conditions imposed, and with given initial conditions of
saturation and saturation distribution of two (or more)
immiscible fluid saturants. The conditions are chosen so
that the energy and mass transport effects are unimportant,
therefore, we can limit attention to momentum transport phenomena.
In particular, we seek a way for describing (predicting) the
course of events during the ensuring unsteady-states marked by
saturation changes, as a function of time and position,
throughout the porous medium system.

Without serious loss of generality, the precise statement
of the problem can be greatly simplified at the onset, as
follows:

1.   By choosing an isotropic porous medium in the sense
that the porosity and permeability (i.e. the interstitial
geometry) are homogeneously uniform throughout the space
of interest (at least in all flow directions), and by
saying that the internal pore surface is continuous and
chemically the same at every point.

2.   By stipulating that the solid matrix itself is non-
porous, rigid, incompressible, insoluble and inert.

3.   By having only two fluids as the pore saturants,
one of which is taken as the wetting liquid (that is,
the equilibrium contact angle measured through this liquid
is less than 90°), and by saying these immiscible fluids
are Newtonian, each being a single component phase
mutually saturated with the other, and each having
negligible compressibility but having comparable (say,
equal) viscosity and density.

4.   By limiting attention in the macroscopic sense to
one-dimensional flow (e.g. the source and sink are the
opposite faces of an otherwise encased cube), where the
Reynolds Number is low enough (say, less than unity)
so that turbulence is avoided and so that the nonlinearities
in streamline flow will be minimized.

Obviously, the system under consideration could be con-
structed by taking a macroscopic cubic element of a random
packing of (say, well-sorted glass) spheres as the porous
medium, and by taking (say) pure water and some selected hydro-
carbon fraction as the fluid pair. As a simple condition, we
could let the medium initially be saturated completely with
the wetting liquid, so that our porblem is to trace the history
of what happens when the nonwetting fluid is forced to displace
and replace the wetting liquid (drainage), and subsequently to
trace the history of what happens when the wetting liquid is
brought aack into the porous medium (imbibition). The latter,
of course, can occur spontaneously (via the action of capillary
forces, with or without additional press,re forces externally
imposed). In any case, experience shows* that repeated cycles
of drainage and imbibition displacement give rise to hysteresis
effects of interest, and we shall want our theory to account

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*cf. Scheidegger, loc. cit.

for this irreversibility feature.

Much of the pertinent work on the problem at hand is cited by Scheidegger (1960). The first attempted quantitative statement was given by Richards (1931), but many other investigations have been important in the evolution of the concepts which now are to be examined (cf. Buckley and Leverett, 1942; Brinkman, 1948; Klute, 1952; Philip, 1957; Cardwell, 1959; Blair et al, 1958; Hovanessian and Fayers, 196).

The analyses necessarily starts with a mass continuity statement of the type:

$$(2.1) \qquad \mathrm{div}\ (\ \rho_i\ v_i) = -P \frac{\partial (S_i\ \rho_i)}{\partial t}$$

from which can be derived (Blair et al, 1958):

(2.2)

$$\frac{\partial}{\partial x}\left[\frac{k_w k_m \frac{\partial P_c}{\partial S_w}}{k_w \mu_o + k_o \mu_w}\ \frac{\partial S_w}{\partial x}\right] + \left[\frac{Q}{1 + \frac{k_o \mu_w}{k_w \mu_o}}\right]\frac{\partial S_w}{\partial x} = P \frac{\partial S_w}{\partial t}$$

on assuming that Darcy's law applies, and that the effective permeability and the capillary pressure are known functions (e.g. empirically) of saturation according to:

$$(2.3) \qquad k_i = k_i\ (S_i) = -\ \frac{v_i\ \mu_i}{\mathrm{grad}\ p_i}$$

and

(2.4) $p_c = p_c (S_w)$ where $p_c = p_n - p_w$

Equation (2.2), when combined with (2.3) and (2.4), implicitly has the wanted solution:

(2.5) $S_i = S_i (x,t)$

and thus, in the common view (cf. Hovanessian and Fayers, (196/), it represents "....the fundamental differential equation.../~ [whose] ....solution gives a complete description of the physical problem being studied...."

In the statement of (2.2) to express linear displacement in the absence of gravity, however, the following conditions and conventions have been imposed, namely:

(2.6)
$$\begin{cases} v_w + v_n = \text{const.} \\ p_c = p_n - p_w \\ S_w + S_n = 1 \\ \partial p_i / \partial y = \partial p_i / \partial z = 0 \\ \rho_i = \text{const. and } \Delta \rho = 0 \\ \mu_i = \text{const. and } \Delta \mu = 0 \end{cases}$$

Thus, it is seen that the solution (2.5) is had when the saturation is known as a function of time and position throughout the system, which means that, implicitly, the other dependent variables (v_i, k_i, p_i, p_c) also are determinable and known in terms of x and t, while all other parameters are given constants of the system.

In the most general case of three-phase saturated media, we have in addition to the four independent variables (x,y,z and t), 18 dependent variables (namely, three each of v_i, p_i, k_i, S_i and ρ_i, and three capillary pressure parameters for the 12, 13 and 23 fluid parrings); therefore, we need a set of 18 simultaneous equations each of the continuity statement, the equation of

state, Darcy's law. the $K_i(S_i)$ function; and the capillary
pressure definition; two equations giving $p_c(S_i)$; and one
equation of the form $\sum S_i = 1$).

Indeed, claims have been made (Douglas _et al_, 1959)
that more general solutions can be obtained of the form:

(2.7) $S_i = S_i (x,y,z,t)$

and, to the extent that machine programming and the supple-
mentary experimental work (to give the functions 2.3 and 2.4)
can be undertaken with reasonable labor, ultimately it may be
possible to treat multi-dimensional flows with three or more
(miscible and/or immiscible) fluids present*, taking gravity
as well as capillarity into account, for transport when con-
ditions opposite to (2.6) apply, in porous media which are
neither isotropic nor homogeneous. Consideration of these
multiple possibilities, however, adds nothing to our present
analysis, except thereby we shall draw the inference that it
is self-defeating to increase the order of complexity of a
problem until the elementary solutions have been fully evaluated.

In point of fact, it is our immediate purpose to see in
(2.2) is a proper representation of the dynamics of displacement,
that us, whether or not one can avoid the indeterminacy of
a partial differential equation with three dependent variables
(i.e. 2.1) by reducing the number to one (i.e. by operating on
2.2 with 2.3 and 2.4). It is only in this way that the solution
(2.5) can be obtained, therefore, the availability and accessability
of solutions of the type (2.7) is a contingent matter.

It is necessary, in consequence, to examine in detail the
meaning and applicability of Darcy's law (2.3) and the capillary
pressure function (2.4); for, if these intermediate steps are

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*In miscible fluid displacement systems, a concentration term
is introduced for the saturation term of (2.1), and a diffusivity
parameter replaces the porosity-permeability function applicable
to immiscible fluid displacement.

rejected in part or in whole, the hope of obtaining even
approximate solutions of the type (2.5) and (2.7) is corres-
pondingly diminished. First, however, it is important to con-
sider the quasi-instability matter which gives rise to the
development and growth of so-called "viscous fingers" when the
displacing fluid has a higher mobility than the replaced fluid.
The latter feature often is treated as a separate question
(Scheidegger and Johnson, 1960), but it is evident that viscous
fingering is an important aspect of the unsteady-states of
miscible as well as immiscible fluid displacement processes.
Our phenomenological theory immediately follows.

3.   Instability Considerations

If instabilities occur, the continuity implied by (2.1)
still applies; therefore, we must examine how the instabilities
will affect the form of the solution (2.5).

Let us suppose that Darcy's law (2.3) describes the flow
transport of each fluid pahse, even though there may be a high
viscosity contrast between phases, and even when saturations
are changing rapidly with time. We take this latter as a
simplifying assumption, tentatively, until the matter is dis-
cussed in greater detail in $\oint$ 4 to follow. In this way, we
are free to discuss the criteria which give rise to the onset
and growth of the viscous fingers (which are the manifestation
of the instability), and to employ the convenience of the Darcy
law concept in formulating the problem heuristically.

We introduce first the mobility proportionality factor
as a measure of the ease with which a fluid phase can be forced
to move through a porous medium. Obviously, fluid mobility
is a function of the saturation and the saturation distribution
configuration. In the Darcy law frame of reference, the mobility

can be expressed as an effective permeability to viscosity quotient, k_i/μ_i, dependent on time and on the space co-ordinates, as well as on the past-history (hysteresis of saturation change) of the system in a way which corresponds to the meaning of k_i. A mobility ratio therefore can be defined arbitrarily as:

(3.1) $M = (k_2/k_1) \, (\mu_1/\mu_2)$

where the convention is adopted that fluid 1 is the displacing (entering) fluid, and fluid 2 is the displaced (replaced) fluid.

We now introduce the hypothesis that for horizontal flow (i.e. no gravity forces considered to be acting), the moving flood-front will be unstable for all sensibly finite forward velocities when M is sensibly less than unity. This postulate can be justified qualitatively by considering the instability criteria of Chuoke et al (1958), and by noting that the critical wave-length of flood-front pertubations which must be exceeded if fingering is to ensue numerically is indicated to be sensibly smaller than the wave-length index of heterogeniety (as defined by Scheidegger, 1960). In fact the latter apparantly is true even for what otherwise would be considered as homogeneous media, as will be shown.

Following Smith et al (1933), who introduced the useful concept that a random packing of spheres of uniform size (i.e. a sensibly homogeneous medium) actually has the form of inter-posed asserbleges of blocks where a particular mode of packing is in predominance (e.g. cubic assembleges contiguous to orthorhombic and/or rhombdohedral assembleges, etc.), we suspect that the lower limit for the Scheidegger wave-length index of heterogeniety will be measured by (say) ten to 100 grain

diameters. With it thereby likely that incipient perturbations are a feature of even elementary homogeneous media, therefore, as we consider displacement in less homogeneous (say, heterogeneous) media, for example, where the sorting of particle size and the uniformity of particle shape has been diminished, and/or a micro-bedding trend has been imposed, then the Chouke et al criteria for instability will be met for most practical cases of interest, as long as M is less than unity.

We adopt therefore, as approximately correct, the proposition that M $<$ 1 implies that instabilities will occur during the displacement process under consideration, being prepared to abandon this qualitative statement later whenever the critical value of M is wanted (somewhere between unity and zero) which quantitatively marks the transition between stable and unstable flood-front character.

To facilitate discussion and numerical analysis, albeit without a loss in generality, let us introduce for (2.3):

$$(3.2) \qquad k_w = (S_w - S_{wM})^2 / (1 - S_{wM})^2$$

and:

$$(3.3) \qquad k_n = (1 - S_{nM} - S_w) / (1 - S_{nM} - S_{wM})$$

as approximations to what experiment gives as the usual form of the relative permeability function (cf. Rose, 1949). Then M may be represented by:

$$(3.4) \qquad M =$$

yielding the following critical values of S_w (where M=1) and where the onset of viscous fingering is assumed to occur, namely:

$$(3.5) \qquad S_{w_c} =$$

when $\partial S_w / \partial t < 0$ (i.e. a drainage process)

and:

(3.6) S_{w_c} =

          when $\partial s_w / \partial t > 0$ (i.e. an imbibition process).
where the importance of (3.5) and (3.6) lies in the assumption
that $S_{w_c} > \zeta_w$ for a drainage process and $S_{w_c} < \zeta_w$ for an imbibition

process, corresponds to $M < 1$ (that is, instabilities are to
be expected).

Now, it has been inferred already that (2.5) can be
obtained from (2.2) through substitution of the empirical
functions, (2.3) and (2.4), only when the flood-front is stable
(e.g. $M > 1$). It is important to develop the rationale for
this restriction. The continuity expression (2.1) is always
valid (i.e. "matter can neither be created nor destroyed") quite
apart from any instability consideration; likewise, we are
prepared at the moment not to quibble whether or not use can
be made of the relative permeability function (2.3) and the
capillary pressure function (2.4). The point is that as long
as the flood-front interface is stable, the micro-heterogeneities
of the system can be ignored; and, conversely, when the inter-
face is unstable the micro-heterogeneities suffice to induce
the pertubations which give rise to the observation of the viscous
fingering.

Figure 1 shows the variation of M with S_w as obtained
from (3.4) for different values of the viscosity ration, and
for the indicated sets of assumptions about S_{wM} and S_{MM} chosen
to correspond to the hysteresis in the relative permeability
curves when imbibition and drainage displacement processes are
compared. The shift and relative values of S_{wc} as defined by
(3.5) and (3.6) are of immediate interest.

Let us consider first an imbibition process with the initial
conditions implied by Figure 2. If the wetting fluid viscosity

is considerably greater than that of the replaced fluid, S_{W_C}
will be close in value to $(1-S_{MM})$; therefore, taking x_1 as the
level of observation, S_W will be seen to continuously increase
at x_1 as the flood-front passes under displacement conditions
where M is favorable for most of the time. Indeed, no instability
can occur until S_W exceeds S_{W_C}, but time-wise this does not

happen until most of the nonwetting fluid has been displaced
from the level of x_1, which is to say that the displacement
efficiency of the process will be negligibly affected by any
viscous fingering which may ensue due to the fact that k_N is
approaching some vanishing value.

On the other hand, for imbibition displacement processes
where S_{W_C} is close to S_{WM}, the onset of instabilities at the
flood-front will be observed at the level x_1 almost immediately,
and a corresponding reduction in the displacement efficiency
of the process can be expected.

The final cases to be considered have to do with drainage
displacement (i.e. where the nonwetting fluid replaces the
wetting fluid), and it will be noticed again from Figure 1 that
the proximity of S_{W_C} to S_{W_M} (favorable), or to $(1-S_{MM})$
(unfavorable), is the factor which determines at what stage
of the displacement process the instabilities are manifested.

Thus it is seen that when the displacing fluid is con-
siderably more viscous than the replaced fluid, fingering is
deferred until the last stages of displacement; therefore, the
instability phenomena will not be noticed beyond (possibly)
the observation of a slightly higher "irreducible" saturation
of the replaced fluid than that limiting value attained after
extremely long periods of extremely slow rate-of-flood-front
advance. It is problems of this sort for which Equation (2.2)

has been developed by previous workers as the appropriate
description, and for which solutions of the type (2.5) have
been obtained by making use of the empirically determined
functions (2.3) and (2.4).

4.   Summary Statement

In lieu of the presentation of the intended remaining
part of Section 3 above, and of Sections 4 and 5 referred to
in the introduction to this paper*, an outline of the argument
is summarized below.

The general theory of immiscible fluid-fluid displace-
ment must account for the growth and development of viscous
fingering, and it is likely that appeal can be made to in-
stability considerations to obtain a phenomenolical description.
In the more basic sense, however, a single unifying representation
must exist which is general for the case of an "unfavorable"
as well as a "favorable" viscosity ratio. This paper seeks
to find that unifying representation, by considering the dynamics
of displacement as viewed with respect to the microscopics of
the system. The latter is made necessary, in addition, because
of the considerations previously brought forth by Rose and
Channapragada (cf. also Massé, loc. cit.) dealing[1] with the
uncertainties implicit in the use of Darcy's law to describe
the unsteady-states of mixture flow, and[2], with the fact that
the capillary driving forces have their source in microscopically
distributed interfaces.

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*to appear later.

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NETWORKS REPRESENTING POROUS MEDIA FLOW SYSTEMS
By
Walter Ross

Abstract*

We limit attention to cases of streamline flow of
essentially incompressible fluid(s) being transported
through porous media systems which are characterized by
varying degrees of inhomogeniety and anisotropy, however,
both steady and unsteady flow situations are to be considered.

As a foundation, we then state (in a brief but rigorous
fashion) the consequences of applying Navier-Stokes theory
to give a description, in microscopic detail, about the
motion of fluid particles through interstitial space. These
fluid particles are identified as microscopic differential
volume elements each assemblages of a statistical number of
momentarily contiguous fluid molecules) to which locations
must be assigned in space as a function of time and which
are seen to trace the streamline paths between sources and
sinks.

We find it instructive to describe in qualitatively
the tremendous complexity of what actually happens during
flow transfer in porous media systems, for out of this we
are led to a new derivation of Darcy's law from Navier-Stokes
theory which avoids the suspect features of previously
offered derivations (Hubbert, Philip, Hall, Miller and Miller,
etc.). More important, our analysis has the advantage of
identifying, a priori, systems and conditions where use of
Darcy's law can be expected to fail in describing the flow

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process under investigation.

In the end, we wish to treat porous media flow systems according to the methods of potential theory, but this representation is possible only to the extent that we can assign macroscopically meaningful values of a permeability tensor to volume elements of the system large enough to contain a "statistical" amount of pore space, but small with reference to the relevant gross dimensions of the systems of interest. We must also be able to say that a field of flux and a field of force can be constructed, which in some way approximate a one to one correspondence to the values of velocity vector and potential of the microscopically defined fluid particles. We thus construct an analog space, each point of which represents discrete volume elements in real space, and we regard this analog space as a continuous media.

We find that a continuous media representation of porous media flows cannot be undertaken in all cases, even when attention is limited to the laminar motion of incompressible fluids.

In the microscopic frame of reference, there will be continuity of streamline paths where these will be defined as the trace of fluid particles (i.e. a microscopic differential volume element which is an assemblage of a statistical number of momentarily contiguous fluid molecules) joining sources to sinks. The paths may not be fixed in space with time, but intersections leading to the stagnation points characteristic of certain inviscid fluid flow systems will not be encountered. The fluid particles themselves will, in general, undergo successive (and perhaps time-variant) acceleration-decelerations, leading sometimes to a dispersal which must be taken into account

if molecular diffusion is to be represented. To each
fluid particle may be ascribed a certain value of potential,
kinetic and pressure energy, as well as a certain value of
internal energy (including surface energy if the fluid particle
is located at an interface). In the limit, the pore wall
boundaries are the loci of the zero-velocity streamline paths,
which is to say that the adsorbed fluid particles there have
zero kinetic energy.

Continuous matter theory deals with the representation
of the properties of matter as these are observed in, and
defined by reference to assemblages of a large (statistically
sufficient) number of contiguous molecules. Such assemblages
themselves can be quite small, as illustrated by the fact that
the density (or other property) of a particular material,
held everywhere at the same temperature and pressure, will
be the same as a sample is subdivided (say) below the limits
of visual resolution. The subdivision cannot be carried with-
out limit, however, because of the particulate nature of
matter, and as the volume element dimensions approach molecular
sizes, macroscopic properties of matter become difficult to
define, much less measure.

In this essay we do not concern ourselves with what
gives rise to particular properties of matter, that is, we
make no appeal to kinetic theories nor to statistical mechanics.
We choose simply to postulate that we have a continuous medium,
bounded or unbounded, where at every internal point a property
such as "demittance"* can be defined and measured.

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*Demittance refers to any definable and measureable property
of matter whose definition and measurement are left unspecified
in a particular discussion, to avoid the introduction of any
bias which would come from the discusser's preconceived notion
about the property. The concept and the term were proposed
by L. deWitte, Hughs Aircraft Corporation, 1961.

NETWORKS REPRESENTING POROUS MEDIA FLOW SYSTEMS

I. FOUNDATIONS

By

Walter Rose

Introductory Concepts

In the end we shall be talking exclusively about fluid flow in porous media systems, but much of what will be said has a bearing on mass and energy transport phenomena as well as on momentum transport; therefore, we introduce the elementary concepts in a quite general fashion.

We first discuss some sort of a continuous conductor, without specifying whether it be solid, liquid or gaseous, and whether or not it be homogeneous or isotropic. If conduction is to occur, a driving force must be acting, whic. is to say that a finite energy gradient (e.g. in thermal and/or potential and/or chemical energy, etc.) will be found directed along paths joining the sources to the sink boundaries; moreover, there must be no zero-conductivity barrier(s) completely isolating all sources from all sinks. The conductor may be thought of as an infinite medium if the phenomena of interest occur (during the time interval of interest) for the most part in internal regions far removed from the influence of external boundaries; otherwise, the external geometry must be a specifiable boundary condition.

Our problem is to seek ways to describe and discuss the transport phenomena, given$^{(1)}$ a particular conducting medium having a particular geometry,$^{(2)}$ a particular connection to sources and sinks of given strength, and$^{(3)}$ a particular initial condition of energy distribution. In general we shall find it necessary to consider unsteady as well as steady states, and we shall have to take into explicit account the influences of the inhomogenieties (which say that contiguous points may have different conductivities) and the anisotropies (which say that the conductivity at particular points may depend upon direction).

Because of the particulate nature of matter, it is, in the final analysis, a fiction to speak of a continuous conductor medium. On the other hand, as long as one is dealing with the representation of the properties of matter as they are observed in, and defined by reference to assemblages of a large (statistically sufficient) number of contiguous molecules, all functions of these properties may be thought of as continuous throughout the medium. Therefore, we avoid molecular considerations by taking the macroscopic view that the microscopic volume elements, each comprised of a large number of molecules, may each be thought of as a point in space. The infinite ensemble of these points (occupying either a finite or an infinite space), then, is our continuous medium.

Our first step, therefore, must be to assign values of local conductivity to each point in space, for this will lead to an evaluation of the effective conductivity of the medium, and

to a prediction of what transport will occur for given initial
and boundary conditions. What is implied by this statement is
that we have a partial differential equation to solve, where the
dependent variable(s) of interest (flux, potential) are functions
of both time and position, and where the conductivity of the
medium (at each time and position) explicitly appears as a
parameter.

Indeed, the differential equations which are to be
solved, in the most general cases, are nonlinear, of high order,
and perhaps inhomogeneous; therefore, great problems of analysis
are met with from the start. And still to be taken into considera-
tion are the added complications provided by the stipulation of
unsymmetric and/or time-variant boundary conditions, the choice of
complex initial conditions, and the circumstance that the medium
may be neither homogeneous nor isotropic.[+]

We choose now to simplify the problem by saying that
the given continuous medium can be subdivided into a three-
dimensional array of a finite number of discrete blocks. We then
seek, for example by relaxation methods, to describe the inter-
action between each contiguous element. In effect, we hope to
come to numerical results by the traditional procedure of
substituting finite differences for infinitesimal differentials,

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[+] The suggestion that a differential equation has to be solved
rests on the assumption that a satisfactory basis for abstraction
has been found whereby a mathematical model (likely more imperfect
than perfect) has been devised to represent the physical system
of interest.

and by discarding the notion of a truly continuous conductor medium in favor of a more manageable network form. That is, each block of the network will have lumped in it the appropriate characteristics of the volume element it represents (in accordance with some averaging process still to be defined), in a manner which is equivalent to saying that the surfaces which are common boundaries to contiguous volume elements in real space are represented in the network as points of contact. As a consequence, we shall reduce the degrees of freedom of the problem, so that by successive approximation we hope to converge on a satisfactory solution.

To restate in another way what is to be undertaken, we are saying that a mathematical point in real space is a "physically" infinitely small volume, or, as described by Landau and Lifshitz (Fluid Mechanics, Volume 6 of Course in Theoretical Physics, Pergamon Press, 1959, Chapter 1), a volume element particle "...very small compared with the body under consideration, but large compared with the distances between molecules...". Thus, we may expect in some cases that the microscopic differential volume elements must be chosen say, smaller than a cubic micron in order to think of a particular medium (say, one of quite small dimensions and quite inhomogeneous) as a continuous conductor. Alternately, another (say, larger and more homogeneous) particular media may display all the characteristics of a continous conductor, even when volume elements having dimensions measured by orders of cubic meters are taken as the "points" in mathematical space. The network model concept, therefore, is derived from the idea that the interaction between two adjacent collection of contiguous points can be approximated by noting the interaction between two other points, each of which has the "average" properties of one or the other of the collections.

The Network Model

Let us first address ourselves to questions of potential flows. During the laminar transport of phases (as in fluid flow), components of phases (as in diffusion), or energy (e.g. heat or electricity) through conducting media, a so-called field of flow will exist which will be representable as paths joining sources and sinks along which conduction occurs. The trace of these paths will be determined by the inhomogeniety of the medium, by the location of sources and sinks and by the geometry of the other boundaries, and by the field of force which depicts the action of the force-intensity vector. If anisotropies exist the field of flow does not superimpose on the field of force; moreover, the flow lines may depend on the initial conditions of energy distribution if unsteady-states ensue, and may gradually shift in position.

In order to approximate the action of a continuous conductor by a network model, therefore, one must choose a form and a grid spacing such that the flow lines of the network are a close (step-wise) representation of the flow lines of the prototype. Therefore, as a rule of thumb, coarse spacings of network elements will be a reasonable approximation when the medium being represented is homogeneous and isotropic, and for when attention is limited to steady-states, and/or when the boundaries are such that a single network form can be constructed which has the configuration closely superimposing on the actual

field of flow; otherwise fine spacings are required, which is to say that the model must represent a high degree of subdivision of the continuous conductor media prototype.

The foregoing statements, of course, suffer in exactness from their own relativity, but the point is clear that in many practical cases it will be possible to assign upper and lower limits to the number of network elements, optimized so that resolution and convergence on the sought answer will be maximized, while the computational labor will be minimized. Indeed, it is the major purpose of our discussion to examine how to arrive at a decision about network size and form, and to consider how much will be lost in accuracy of representation in the effort to avoid an unreasonable amount of numerical analysis.

Figure 1 shows one plane of a cubic network form.[+] Each contiguous node (grid point, mesh point, etc.) is to be thought of as representing the center of volume of contiguous macroscopic volume elements (ΔV) in the prototype; therefore, the number of nodes, N, is a measure of the degree of subdivision which is intended. The nature and the size of the elements surrounding each node, then, are taken to represent the relevant average properties of the prototype medium in the volume element which each node point depicts.

[+] A wide variety of two and three dimensional network forms may be imagined, which can be regular or irregular. In particular cases, however, the geometries may be such that it may prove sensible to choose a particular network form to maximize the overlap of the flow paths in the model and the prototype systems for a given value of N. On the other hand, we shall limit reference to cubic forms as they are easy to discuss, and in fact always can be used if N is large.

In effect, we started with a notion of a continuous medium, and took the (usually) safe position that a statistical collection of molecules is "small" enough to be representable as a mathematical point. The sense of constructing a network model is that we now have taken a statistical collection of contiguous points (i.e. the microscopic differential volume elements, dV, each of which contains a statistical number of molecules), and we call this macroscopic differential volume element, ΔV, also a point (i.e. a node point) in the network model space. Thus, we have constructed, in network form, a new quasi-continuous medium, the contiguous points of which represent a sensibly finite distance in real space.

It follows, that in transforming from real space to network model space, we gain little or nothing in the analytic formulation of the problem, even though the exact initial and boundary conditions and the exact local conductivity of the prototype system, in general, are simplified (i.e. roughly approximated) in the network model analog.[+] Thus, Figure 2 depicts how a given network form and spacing can only approximate, but not match the conditions and performance of a particular conduction system.[++] But in spite of the loss of resolution when the problem is cast into its network analog representation, the exact solution of the network model problem involves a similar

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[+] Elsewhere we discuss at length how, in the case of fluid flow through porous media, a simplicity is introduced in the fact that a more tractable law of force applies to the network model system (i.e. Darcy's law) than to the microscopic prototype system (i.e. the Navier-Stokes set of equations).

[++] The prototype flow path of Figure 2 meanders in a way related to the inhomogeniety and anisotropy of the system, as well as to the imposed initial and boundary conditions.

analytic formulation as characterizes the solution of the prototype problem. In both cases, conservation of mass and energy must be established for every point in the system (i.e. the continuity statements), and an equivalent law of motion must be stated and applied to every point in the system. This means, in addition, that the conductivity tensor must be known at every point, whether we are talking about the prototype system or the network analog thereof.

On the other hand, and as to be expected, advantages do result by transforming a conduction problem into its network representation. In the first place, if the prototype system is characterized by a high degree of inhomogeniety, explicit solutions are not to be expected even for cases of simple initial and boundary conditions. Similarly, if the prototype is quite homogeneous[+] (whether locally there is isotropy or not), explicit solutions are available, if at all, only for cases of simple initial and boundary conditions. It will be seen, therefore, that the gain achieved by reducing the problem to its network model form is three-fold. If an analytic solution is possible at all, one immediately gains the advantage that conditions and

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[+] We define homogeniety to describe a medium where the conductivity tensor at all contiguous points will be the same, or nearly so, and will have the same **configuration.** In consequence, many types of inhomogeniety (i.e. heterogeniety) may be imagined, including bedded media, graded media, media where the inhomogeniety comes only from the fact that there is an irregular orientation of anisotropic contiguous elements, etc. In any case, it is clear that any network model will tend to be less inhomogeneous than the prototype from which it was derived, and will tend to have an altered anisotropy character. In a corresponding sense, it is to be noted that, in the limit, a medium which is completely inhomogeneous in the microscopic sense (by the definition given above) would have the outward (macroscopic) ... a ... ... completely homogeneous

characteristics need be specified only at a limited number of points (i.e. at the N nodes), instead of continuously throughout space. Secondly, if the number of nodes is not too large, closed (implicit) solutions, involving the simultaneous solution of some N equations with N unknowns, are practical to employ. Lastly, network problems are by their formulation already reduced to the point where standard relaxation and other (explicit) numerical methods (cf. for example, the Hypercircle method of Synge) can be applied.

Network Model of Continuous Media

We choose now to illustrate some points by making reference to a three dimensional heat flow system, for example, that having the geometry of Figure 3, and made up of a large (essentially infinite) number of single crystals of various atomic species. Each crystal will be thought of as being at least as large as whatever is required so that its lattice contains the (so-called) statistical number of atoms; therefore, if this mixture of crystals is pressed together to form a solid mass, we can say we are dealing with a (virtual) continuous medium. Further, we shall say that the medium is inhomogeneous, at least on the microscopic scale, because we postulate that each species of crystal has its own value of specific heat conductivity, σ_{ij} . To be general, we say that some of the crystals undergoing radio-active decay (say, of sensibly long half-life) so that we may think of them as internally distributed heat sources. Other (external) sources and sinks of given strengths are shown in Figure 3A, together with the trace of certain arbitrary flow lines of interest.

We do not let transfer of energy by convection or radiation
enter into the formulation of this problem. Therefore,
to describe heat flow (by conduction) under these conditions,
we need to know the initial distribution of temperature at
every (crystal) point in the system, and the strength (and
its time variation, if any) of every source and sink; moreover,
we must know the thermal conductivity tensor* of each single
crystal point, and we must have the geometry of all surfaces
specified across which there is no flow (i.e. the impermeable
boundaries).

Now, for each point in the system, except on the source
and sink boundaries and the internally distributed source
points , a continuity statement can be applied to the effect
that the divergence of heat flux is proportional to the change
in heat content with time, and Fourier's law of heat conduction
can be applied to give as the governing differential equation
of motion:

$$ \operatorname{div} \left[\sigma_{ij} \, \vec{\nabla \phi}' \right] = - c' \frac{\partial}{\partial t} \phi' = \operatorname{div} \vec{\gamma}' $$

where:

$\vec{\gamma}'$ = Microscopic flux per unit area vector,

ϕ' = microscopic value of potential at (x,y,z,t),

c' $\neq$ microscopic value of local capacitance,

σ_{ij} = local value of conductivity tensor.   Eq 1

To be noted is the fact that Equation 1 is most readily
integrable when steady-states are being discussed (i.e. the
right-hand member is equal to zero, so that Equation 1 reduces

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*It is a consequence of continuous media theory that a symmetric,
second-rank tensor will describe the most general case of aniso-
tropy (cf. transmission of heat, electricity, electromagnetic
radiation, acoustic energy, fluid flow, diffusion, etc. through
anisotropic media).

to the Laplacian form); but even with this simplification, the imposition of asymetric and/or non-linear initial and boundary condition in most cases will introduce great analytic difficulties.

One device for circumventing analysis problems, and in fact the one which we propose, is to construct an analog network of the given continuous conductor system, as shown in Figures 3B, 3C and 3D. (In ways to be described later, we can determine that Figure 3C represents the optimum model of the prototype system, as the coarse grid sought, and the fine grid of Figure 3D would involve too much computation). These models however, are at best only approximately equivalent to Figure 3A.

In any case, the network model of Figure 3C, in turn, may or may not be analyzable by exact methods, largely depending on the number of node points, N, involved, and to a lesser extent depending on symmetry and homogeniety considerations. In order to anticipate the difficulties of the most general case, however, let us assume that analytic solutions are impractical* or impossible.

We now assume that a flow equation analogous to Equation 1 can be stated which will apply to the node points of the network model. Since the latter arbitrarily has been put in a cubic form, we choose the coordinate axes to correspond to the principal directions of the network, giving:

$$\sigma_x \frac{\partial}{\partial x} \phi_x + \sigma_y \frac{\partial}{\partial y} \phi_y + \sigma_z \frac{\partial}{\partial z} \phi_z = -c \frac{\partial \phi}{\partial t}$$

Eg 2

---

*Our high speed, digital computer, ILLIAC, in general cannot handle more than 140 simultaneous equations of 140 unknowns. This refers to linear polynomial matrices.

It is thus implied that the terms of Equation 2 are to
be thought of as averages obtained from the terms of
Equation 1 according to the definitions:

$$\vec{v} = i\, v_x + j\, v_y + k\, v_z = \frac{\int \vec{v}\,' \partial V}{\int \partial V}$$

$$\nabla\phi = i\,\nabla\phi_x + j\,\nabla\phi_y + k\,\nabla\phi_z = \frac{\int \nabla\phi\,' \partial V}{\int \partial V}$$

$$\phi = \frac{\int \phi\,' \partial V}{\int \partial V}$$

$$C = \int c'\,\partial V \qquad etc.$$

$$Fig\ 3$$
$$B$$

Now it will be clear that, as N approaches infinity,
Equations 2 will give the same result as would be obtained
by an analytic solution (whether known or not) of Equations 1,
at least to the extent that the successive approximations
taken in connection with Equation 2 leads to sensible convergence.
Therefore, the question at hand is what value of N shall be
chosen which will give the numerical accuracy of results
which is wanted without entailing an unreasonable amount of
computational labor?

Several reasons, arrived at intuitively, can be cited
why the network form of Figure 3B is less satisfactory than
that of Figure 3C in representing the prototype system under
discussion (cf. Figure 3A). We wish to achieve a high degree
of superposition between the fields of force and flux in the
prototype and model systems.

FIGURES 1 and 3 will be supplied at a later date.
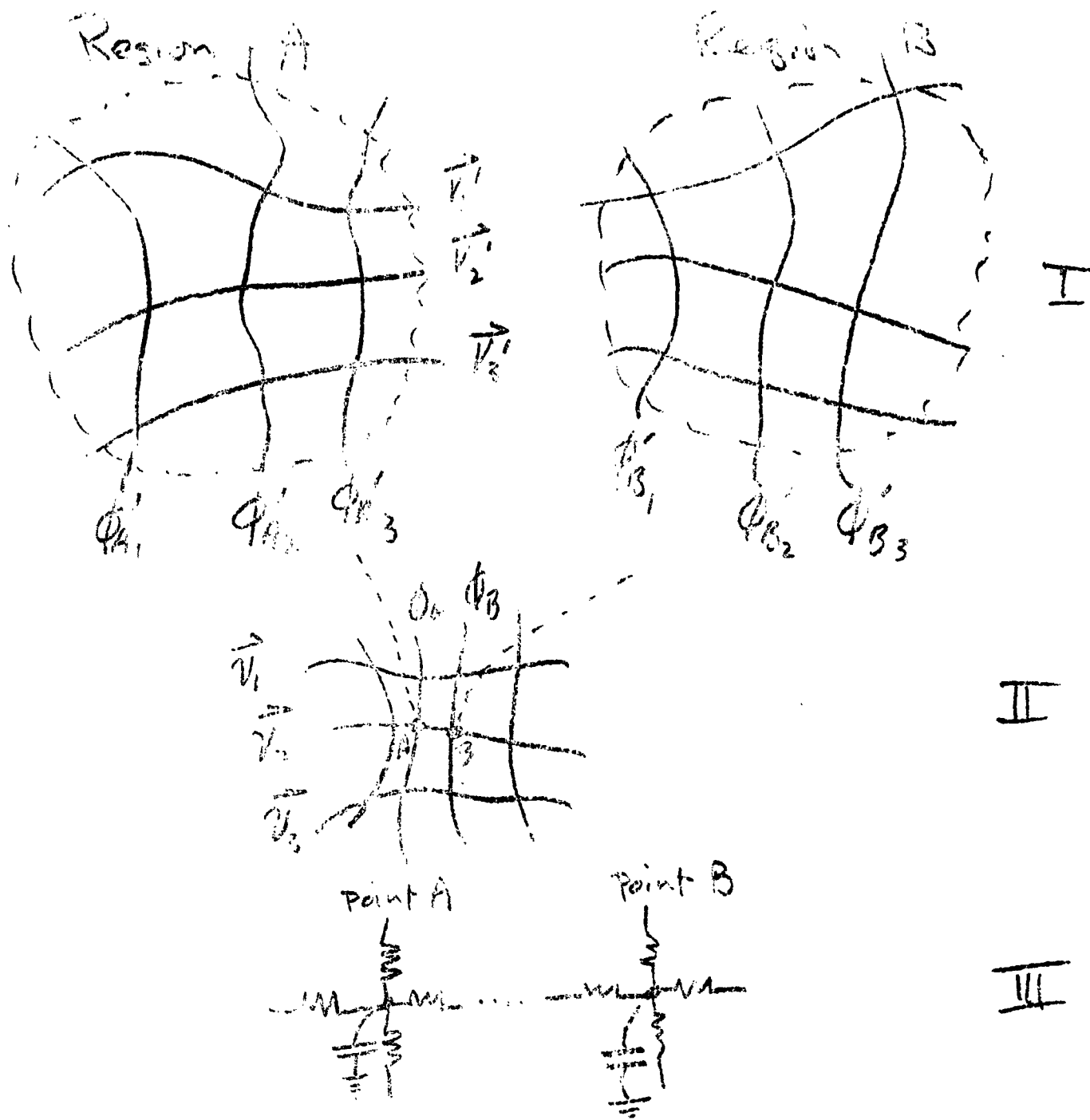
I — Prototype = real (microscopic) space, where within the regions A and B, fluid particles have local and time-dependent values of potential, ϕ'_i, and of the velocity vector, v'_i.

II — Model = quasi-continuous space made up of contiguous macroscopic lumps of real space

III — Network model form of II.

FIGURE 2

GENERALIZED FLOW PROBLEM IN POROUS MEDIA

by Rao Channapragada

Introduction

The two-dimensional flow problem, with a denumerable number of arbitrary shaped grains in a pore channel, is analyzed analogous to the treatment of P. B. Richards[1] and some applications of the author's doctoral thesis[2] have been introduced.

As pointed out by Richards each of the contours of the grains can be replaced by a sheet of vortex-like singularities such that the velocity induced by each element of the sheet satisfies the boundary conditions of the flow on the channel walls. The complex velocity on the contours satisfies the singular integral equation with Cauchy-type kernel taken over the contours alone. This can be reduced to a linear Fredholm integral equation of the second kind with a continued kernel for the tangential velocity on the contour and the solution of the Fredholm equations follows naturally[3]. The method of attack of the flow problem using singular integral equations offers definite computational advantages and it suffices to quote Richards: "Previous analytical work on channel flow has been more restricted in its applications and/or complicated computationally."

With no loss of generality let the pore channel be an infinite strip of width r and $-\frac{r}{2} \leqslant Y \leqslant \frac{r}{2}$. Let the free stream velocity of the flow in the positive x-direction be $V = q_x(\infty)$ and $U_\nu(z,r)$ ($\nu=1,2,\ldots n$) is the complex velocity induced by the entire vortex sheet coinciding with the contours of the grains C_ν.

Let $f(z)$ be the complex potential of the vortex-like singularity located at $t = \xi + i\eta$ on C_v.

We now seek an analytic solution to the channel flow in the form

$$w(z,r) = V + U_v(z,r) \qquad U_v(x = \pm\infty) = 0 \tag{1}$$

The induced velocity corresponding to C_v

$$U_v(z,r) = \frac{1}{2\pi} \oint_{C_v} k(t,r) \frac{dw}{dz}(z,t,r)\, dt \tag{2}$$

where the complex potential of a unit vortex at $t \in C_v$ is given by

$$w = -i \log \frac{\exp(\pi z/r) - \exp(\pi t/r)}{\exp(\pi z/r) + \exp(\pi t/r)} \tag{3}$$

and the complex velocity

$$\frac{dw}{dz} = i\, \frac{u(t,z,r)}{t-z} \tag{4}$$

where

$$u(t,z,r) = \frac{\pi}{r}\left[\frac{t-z}{\exp(\pi/r)(t-z)-1} \quad \frac{t-z}{\exp(\pi/r)(\bar{t}-z)+1}\right. \tag{5}$$

Therefore

$$U_v(z,r) = -\frac{1}{2\pi i} \oint_{C_v} \frac{k(t,r)\, u(t,z,r)}{t-z}\, dt \tag{6}$$

In equation (1) the condition $V = 0$ on C_v implies $U_v(z,r)$ has jump values at C_v since V is a constant.

We solve for these jump values in $U_v(z,r)$ by making use of the well-known Cauchy integrals.

For the Plemelj formulas to hold along C_v, we have the following hypothesis: If the density function $g(t,z)$ satisfies a Lipschitz condition in z, in the neighborhood of a smooth Jordan curve C_v and also satisfies a Lipschitz condition in t on C_v, then the principal value $G(t_o)$ of the Cauchy integral

$$G(z) = \frac{1}{2\pi i} \oint_{C_v} \frac{g(t,z)}{t-z}\, dt \tag{7}$$

exists for $z = t_o \in C_v$ as well as the limiting values $G^+(t_o)$ and $G^-(t_o)$ as the point t_o is approached from the left $(+)$ of C_v and the right $(-)$ of C_v.

Thus the Plemelj formulas can be written as

$$G^+(t_o) + G^-(t_o) = 2G(t_o)$$

$$\tag{8}$$

$$G^+(t_o) - G^-(t_o) = g(t_o, t_o)$$

Applying these relations to the induced velocity $U_v(z,r)$ which belongs to the class of Cauchy integrals, we obtain an integral equation of the second kind.

Assuming that the vortex strength $k(t,r)$ satisfies a Lipschitz condition on C_v, we have from the Plemelj relations

$$U_v^+(t_o,r) + U_v^-(t_o,r) = 2U(t_o,r)$$

$$\tag{9}$$

$$U_v^-(t_o,r) - U_v^+(t_o,r) = k(t_o,r)\, u\, (t_o,t_o,r)$$

$$U_v(t_o,r) = \frac{1}{2}\left[U_v^+(t_o,r) + U_v^-(t_o,r) \right] \tag{10}$$

and from (1) we have

$$U^+(t_0, r) = V + U_v^+(t_0, r)$$

(11)

$$U^-(t_0, r) = V + U^-(t_0, r)$$

$$\therefore \quad U_v(t_0, r) = \tfrac{1}{2} \left[U^+(t_0, r) + U^-(t_0, r) + 2V \right]$$

(12)

But $U^+(t_0, Y) = 0$ inside C_v

$$\therefore \quad U_v(t_0, r) = \frac{U^-(t_0, r)}{2} - V$$

(13)

From (1), (13), (9) and (11) we have

$$U(t_0, r) = \frac{U^-(t_0, r)}{2}$$

$$k(t_0, r) \, u \, (t_0, t_0, r) = U^-(t_0, r)$$

(14)

But $u(t_0, t_0, r) = 1$

Hence the vortex strength is equal to the tangential velocity along C_v, i.e.

$$k(t_0, r) = U^-(t_0, r)$$

Thus upon substitution in (6) we obtain the integral equation

$$U^-(z,r) = ?V - \frac{1}{i\pi} \oint_{C_V} \frac{U^-(t,r)\, u^-(t,z,r)}{t-z}\, dt \qquad (15)$$

where the integral on the right-hand side of equation (15) is the principal value. Thus if the channel flow problem with grain boundaries admits a solution such that velocity $U^-(t,r)$ at C_V satisfies a Lipschitz condition then it is a solution of the integral equation (15) and the velocity field is represented by

$$U(z,r) = V - \frac{1}{2\pi i} \oint_{C_V} \frac{U^-(t,r)\, u^-(t,z,r)}{t-z}\, dt \qquad (16)$$

If the C_V do intersect, i.e. if the grain boundaries are in contact, then a study of the behavior of the singular integral equation (16) at the points of intersection of C_V is to be made.

The analysis for the point of intersection $t^*(\in C_V)$ is analogous to the above discussion with the following modified Plemelj formulae :

$$G^j(t^*) + G^{(\mu+j)}(t^*) = 2G(t^*)$$

$$G^j(t^*) - G^{(\mu+j)}(t^*) = g(t^*,t^*)$$

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1. P. B. Richards, "Representation by vortex-like singularities of obstacle flow in a straight channel", SIAM Vol. 2, June 1961.

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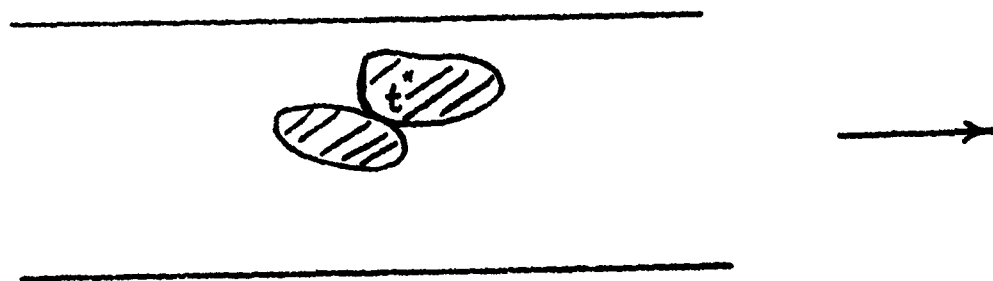## Fig. 1
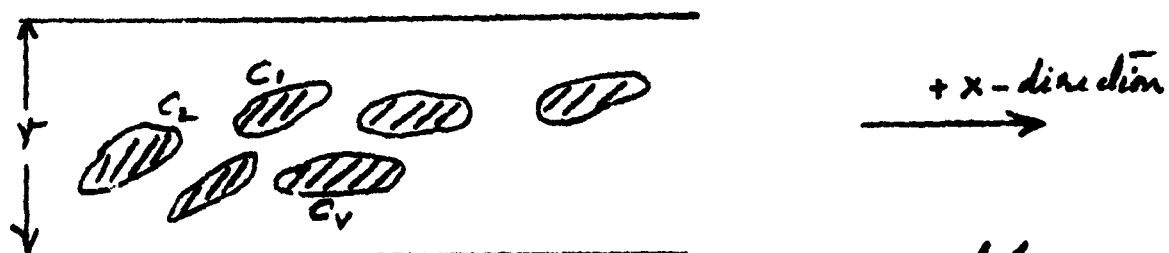
channel flow with denumerable grain boundaries.

Fig. 2

channel flow with grain boundaries in contact

Final Report
By
Rao Channapragada

Summary of Army Chemical Warfare Project CML-517
"Analytical Study of the Behavior of Fluids in Porous Solid Media".

The author was officially appointed to the contract as a research associate from February 1, 1960 to carry out the analytical phase of the study.

The following is an itemized account of the progress made by the author during the period of his appointment.

1). Some basic concepts and definitions with regard to the geometric properties of minimal surfaces and constant curvature surfaces were defined.

2). In the study of immiscible flow through porous media a general definition of capillary pressure and the equations of motion(from microscopic viewpoint) were arrived at.[1]

3). During the study of surfaces of discontinuity the writer was led to the study of singular integral equations. This resulted in the completion of the doctoral thesis of the author[2]. The thesis however was purely from the mathematical standpoint as it was submitted in the Department of Mathematics.

4). An application of the author's thesis has been indicated in the report[3] along the lines of P. B. Richards.

5). An analysis of the flow problem from the fluid dynamic standpoint was initiated and a qualitative analysis is included[4]. This analysis can be further improved from

the mathematical standpoint. However, with the
aid of the physical picture as presented in the
report[4], it is believed that the mathematical
analysis will now be facilitated to a great extent.

Recommendations for further study:

a) It is strongly advisable to confirm experi-
mentally the actual flow pattern of the
streamlines and the presence of the moving
vortices as predicted by the analysis[4].

b) There is a singularity at a point where the
interface meets the wall of the capillary tube.
If the moving interface does not change its
shape the boundary condition at the point of
contact of the interface and the wall is to be
justified since there is a sudden jump in the
magnitude of the velocity (from zero to a finite
value).

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    Media" by Rao Channapragada and Walter Rose, Jour.
    of Geophysical Research, April 1961. Vol. 66 #4.

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    intersecting areas" by Rao Channapragada. Doctoral
    thesis, Department of Mathematics, University of
    Illinois, June 1961.

3). "Generalized flow in Porous Media" By Rao Channapragada.
    (See previous section).

4). "Two dimensional Analysis of an immiscible flow
    through a capillary tube" by Rao Channapragada.
    (See next section).

TWO DIMENSIONAL ANALYSIS OF AN IMMISCIBLE FLOW THROUGH A CAPILLARY TUBE
By
Rao Channapragada

Let the capillary tube be placed horizontally and the
pressure gradient be the force causing the flow of two immiscible
fluids across the tube. The flow is assumed to be incom-
pressible viscous and laminar. It is well known[1] that the
interface (surface of discontinuity between the two fluids)
moving across the tube satisfies Laplaces condition

$$\left(\frac{1}{R_1} + \frac{1}{R_2} \right) = K$$

where R_1 and R_2 are principal radii of curvature and K is
assumed to be a constant. During the flow it is assumed that
the interface does not change shape once the flow has been
stabilized (the pressure difference across the tube is kept
constant). This has been verified experimentally by Rose
and Heins[2].

For laminar flow the velocity profile (Figure 1) is
parabolic and at the interface itself the velocity at each
point of the interface is the same.

The question that naturally arises is as to how does
the parabolic velocity profile transform itself to that of
the interface (Figure 1).

From the photographs of Foettinger[3] it is deduced that
the adverse pressure gradient together with friction near
the wall determine the process of separation (Figure 2).
The presence of a thin plate along the plane of symmetry at
right angles to the first wall (reference 3, Figure 2.14
and 2.15) gives rise to the two symmetrically placed vortecies.
Now imagine the same vertical wall with thin plate to be set

in motion because of the pressure force acting on the wall.
It can again be shown experimentally that there will still be
two stabilised vortecies moving with the wall (similar to
Figure 2). The strength of these vortecies will be lower than
in the case of the stationary wall because of the decrease in
the relative velocity of the flow just ahead of the vertical
wall.

If instead of the thin plate at the center (Figure 2)
two end plates are placed on the ends of the vertical wall,
we will observe a fountain effect (Figure 2a) where the flow
reverses at the corners and the vortecies are again symmetrical
but now displaced from the center of the moving wall to the
corners.

A similar phenomena takes place in the case of the moving
interface. Here the vertical wall is replaced by the inter-
face and the end plates by the walls of the capillary tube.

Consider the region A-B of Figure 3. The stream lines
separate from the relative stagnation point (moving with the
interface) and two stable vortecies are formed moving with
the interface (Figure 3). Also a secondary flow is set up
due to the formation of the vortecies causing the fluid
particles near the wall to be pushed backwards.

The fluid being viscous and for low Reynolds numbers
the secondary flow effect cannot be carried too far from the
interface as shown schematically by the velocity profiles in
Figure 4. The regions of secondary and primary flows are
separated by the limiting lines as shown in Figure 5.

Now consider the velocity profiles across the interface.
As the interface is moving with a constant velocity, the
velocity vectors on either side of the interface are equal

(in magnitude and direction), (velocity profiles P_1 and P_3)
(Figure 6). That is the flow on either side in the immediate
vicinity of the interface is in the same direction. In the
region BC of Figure 3 there is an inverse fountain effect in
the stream line flow. This is again caused by the presence
of the two vortecies as shown in Figure 3.

The stagnation point for the region BC occurs at a point
between the vortecies and just before the commencement of the
inverse fountain flow, indicated by the dashed lines separating
the secondary flow region from the main flow.

This analysis gives us a physical picture of the flow
phenomena in the capillary tube. The above analysis has
been carried out strictly from physical reasoning and it is
hoped that a mathematical analysis of the above would rein-
force the above analysis.

The idea of the fountain effect was originated from the
discussions with Professor W. D. Rose and the photographs of
Foettinger gave an insight into the physical reasoning of the
above analysis.

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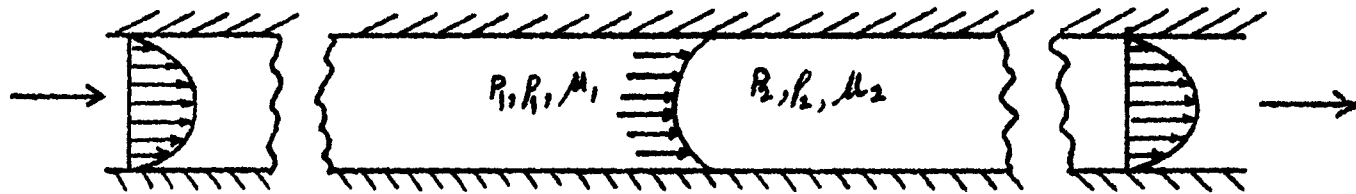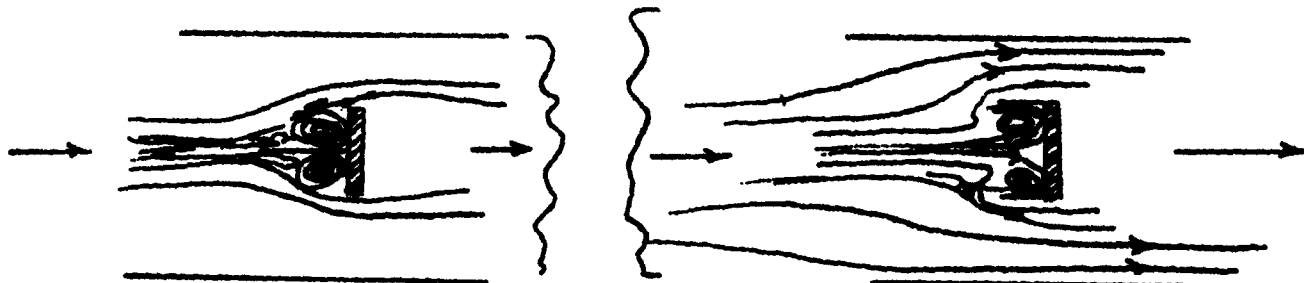
Two dimensional immiscible flow through a capillary tube

Fig.1

(Fig.2)   After Foettinger

(Schlichting Boundary Layer theory R33)

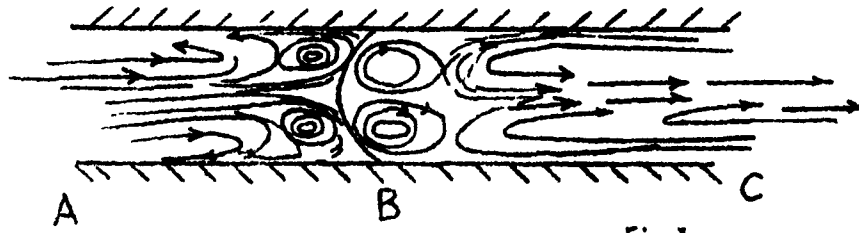(Fig. 2 a)

Fig. 3.

Fig. 4.

Secondary flow region
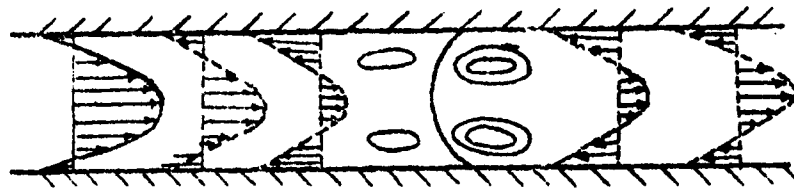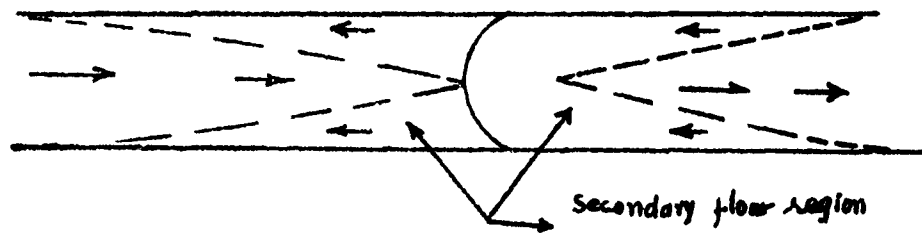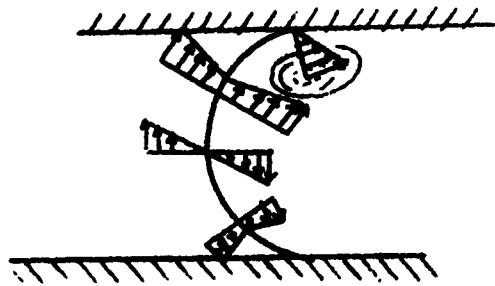
Fig. 5.

Fig. 6   Velocity Profiles at various
          Points accross the interface

ON INITIAL CONDITIONS IN SPONTANEOUS CAPILLARY FLOW

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ABSTRACT

If one end of a capillary filled with air or any vapor
is placed in contact with a   reservoir of wetting liquid,
flow into the tube will be initiated spontaneously and im-
pulsively. An approximate analysis of the subsequent motion
considers the imbibing column as a single (increasing)
mass and yields an ordinary differential equation for the
extent of rise as a function of time. The setting of the
initial conditions (the velocity and position of the interface
between the liquid and vapor) presents some difficulties:
earlier treatments produce some inconsistencies and the re-
sulting integrals do not exhibit the initial impulsive
transient. In this paper general initial conditions which are
physically and mathematically consistent are developed. The
duration of the transient is given as a function of the
initial conditions, and of the surface tension, viscosity
and density of the liquid. An upper bound for the maximum
velocity is derived.

INTRODUCTION

An incompressible wetting liquid of density ρ_L and viscosity μ_L and driven by capillary forces is advancing into a capillary of uniform circular cross-section πr^2 and length ℓ and displacing air or any vapor of low density (ρ_G) and low viscosity, (μ_G). Even for this simple example of capillary flow, an exact mathematical analysis would involve the solution of an intractable two-region (liquid and vapor) boundary value problem consisting of three partial differential equations and appropriate boundary conditions. [1] Further complications arise because the capillary driving force is a function of the momentary fluid-fluid interface geometry, and thus may be time-dependent; and even if compressibility is neglected, it is difficult to set well-defined conditions at this surface since a model for this third 'region' is required [2,3]. An approximate treatment considers the imbibing liquid column as a single (increasing) mass subject to driving and retarding forces; the viscosity and density of the vapor are usually neglected. This macroscopic approach goes back at least to Wolff 4 and has been adopted by many others [5 - 8]. It yields an ordinary, though usually non-linear differential equation for the position of the interface Z(t), that is for the length of the column, as a function of time. The integral solutions vary in detail depending on the conditions (vertical or horizontal, flow, etc.) and on the assumptions made with respect to the forces involved. Newton's law of force applied to the moving column of fluid yields in general an initial value situation:

$$F(z, z', z'') = 0 \tag{1}$$

$$z(0) = z_o \tag{2a}$$

$$z'(0) = z'_o = v_o \tag{2b}$$

where Z', Z'' are the velocity and acceleration. A recent application is that of Siegel [9] who studied spontaneous capillary rise in reduced and zero gravity fields (i.e. for some ambient conditions in space flight). It should be noted that although rise was initiated spontaneously by a sudden reduction in the 'weight' of the liquid, the initial conditions were well defined: viz. the liquid was first permitted to rise to its equilibrium height Z_e under normal gravity and come to rest ($V_o = 0$).

W. E. Brittin [7] among others analyses the dynamics preceeding Siegel's chosen initial situation; that is, one end of a capillary filled with air or vapor is touched to a reservoir of wetting liquid so that flow into the tube is initiated spontaneously. Here, the flow initiation phase is an implusive transient during which molecular forces of adhesion come into play. This would be difficult to analyse especially because of the entry turbulence. Obviously, one should attempt to set physically reasonable and mathematically consistent initial conditions. The present writers in considering the integration of variants of equation (1) by power series, numerical methods and by use of an electric analogue device noted that Z_o and Z'_o as set by Brittin led to physical and mathematical inconsistencies. The sources of these are discussed in the Appendix. An alternate general procedure was developed, therefore, and the results are presented below for an integrable case of equation (1). The duration of the transient is calculated, as well as the maximum velocity attained for all possible values of the viscosity, density and surface tension, dependent on the initial conditions

subject only to the limitations of any particular model
(eq. 1). In consequence an upper bound for the maximum
velocity is derived.

INITIAL CONDITIONS AND THE TRANSIENT

A definite time zero is not obvious when one treats
the initial liquid entry into a (horizontal or vertical)
tube. Let us assume that the contact between tube end and
the liquid is effected under near equilibrium conditions, i.e.
quasistatically. Specifically, let the velocity of the tube
relative to the reservoir of liquid be zero the instant
capillary flow is established. Then the instant when motion
of a column of liquid impends would be an appropriate and
unambiguous time zero reference point, and so by definition:

$$V_o = Z_o' = 0 \qquad (3a)$$

However, before estimating Z_o, we should note the dual aspect
of equation (1). It is a kinematic equation for the position
of the advancing interface, but (more basically) it is an
equation of dynamics for the mass (in units of $\rho_L \pi r^2$). An
effective inertia must be overcome when motion starts, of
magnitude at least that of the mass of the vapor in the capillary
$= \pi r^2 \rho_G \ell$. This quantity is a minimum for the initial mass.
Conversion to the length dimension Z, which refers to the
liquid gives $Z_o^G = (\rho_G / \rho_L) \ell$ to denote that part of Z_o contributed
by the vapor. It is a very small quantity but always larger
than zero. No Z_o^G is a weak lower bound for Z_o since the
effective mass includes liquid above and below the entry end
of the capillary. A better minimum value is calculated by
including the mass of liquid above the bottom of a meniscus
with volume, say, $1/3 \pi r^3$, and by equating this to a cylinder

of height z_o^L with volume $\pi r^2 z_o^L$ from which $z_o^L = r/3$.
Consequently, the total initial mass $= \pi r^2 (\rho_G + \rho_L r/3) = \pi r^2 \rho_L z_o$, from which:

$$z_o = z_o + z_o^L = \rho_G / \rho_L + r/3 \qquad (3b)$$

P = 0 and the conditions (3a,b) must predict an impulsive transient of very short duration in agreement with experience. Initially, there should be a brief period of positive acceleration while the velocity attains its maximum value at which time Z" = 0. This marks the end of the initial phase. Thereafter retardation should predominate with Z" < 0, ultimately increasing to zero at time-infinity.

To check the dynamics let us choose a simple case, a horizontal tube with the fluid column subject only to the capillary driving force and the viscous resistance. Then the time rate change of momentum

$$d/dt(\pi r^2 \rho \, zz') = 2\pi r\sigma\cos\theta - 8\pi\mu \, zz' \qquad (4a)$$

where σ is the surface tension, θ the liquid-solid contact angle. The term $2\pi r\sigma\cos\theta$ is the capillary driving force with θ assumed constant, and $8\mu\pi zz'$ is the Poiseuille steady-state viscous drag. (The subscripts have been dropped, since all quantities now refer to the liquid). Both force terms are approximations since θ is known to be velocity dependent [11] and the acceleration terms of the viscous resistance have been neglected [1]. Turbulence at the entry end may be important in the initial stages. These factors would introduce additional retardation into equation (4a).

Equation (4a) takes the form:

$$(zz')' + Qzz' - P = 0 \qquad (4b)$$

with $P = 2\sigma \cos\theta/\rho r$ and $Q = 8\mu/\rho r^2$. It is convenient to

introduce the dimensionless variables T and W defined by
$t = T/Q$ and $W = 2B/\sqrt{P}$. Equation (4b) then becomes

$$(WW')' + WW' - 1 = 0 \tag{4c}$$

which integrates immediately

$$W^2/2 = T - C_1 e^{-T} + C_2 \tag{5a}$$

The initial conditions W_o and W_o' determine the integration constants so that

$$W^2/2 = T + (1-W_o W_o')(e^{-T} -1) + W_o^2/2 \tag{5b}$$

which is a more useful form for discussion.

The initial states of the system may now be examined for $Z_o' = W_o = 0$ and for a range of values $W_o > 0$. W_o may be as small as we please, and the behavior of the solution as W_o becomes very small is of particular interest for our initial value problem. Figure 1 shows the dynamics characteristic for $W_o > 0$ (the particular case is $W_o = 1$). The acceleration W'' is initially $(1/W_o)$ as the law of motion requires, and then declines to zero marking the end of the initial stage $(T=T_m)$. As noted, it is negative thereafter. The initial velocity is of course zero, then rises to a maximum, $V_{max.}$ at the end of the initial stage and thereafter decays going to zero at time-infinity. The mass of the system is initially $W_o = 1$ while the advance of the interface $\Delta W = W-W_o = 0$; and both steadily increase with points of inflection at the end of the initial stage where $W''=0$.

Of interest is the dependence a) of $V_{max.}$, b) of the duration T_m of the build-up phase and c) of the increment of mass (or advance of the interface) $\Delta W_m = W(T_m)-W_o$ on our parameter W_o. First, the duration T_m is given as function of W_o by the equation arising from the condition $W'' = 0$:

$$e^{-T} - (2 T_m + W_o^2) e^{-T_m} - 1 = 0 \tag{6}$$

Then $V_m(W_o)$ and $\Delta W_m(W_o)$ can be calculated from equation (5b). The three functions are plotted in Figure 2. There is always a build-up stage since $W''(T) = 0$ has a positive solution which $\rightarrow 0$ as $W_o \rightarrow 0$. In the limit the actual velocity is $[(2\sigma \cos\theta)/(\rho r)]^{1/2}$, a definite upper bound defined by equation (4a). This bound is the same for vertical rise.

The drag term due to the turbulence has been given as $\frac{1}{4}(z')^2$ and as proportional to $e^{-t}(z')^2$ [7,8]. An analysis of Langhaar [10] of a similar problem leads to $.64 (Z')^2$. In general for any situation one can write (in the W, T variables) $G(T)(W')^2$ with $G(T)$ some positive function and $G'(T) = 0$. This term destroys the linearity of equation (4a) but must lead to a smaller value for $V(T_m) = V$ max. One result however, follows immediately: it can be shown that the upper bound for V max. i.e. the limiting value as $W_o \rightarrow 0$ now becomes:

$$\left[\frac{2\sigma \cos\theta}{\rho r \{1+G(T)\}} \right]^{1/2}_{T=0}$$

that is, it becomes smaller by the factor $\left[1/\{1 + G(0)\} \right]^{\frac{1}{2}}$.

A representative system is a capillary of 10^{-2} radius containing a liquid of 0.125 poise viscosity, 1 gram/cm^3 density and 50 dynes/cm. surface tension. For these parameters $z \approx W/10^2$, $t \approx T/10^4$. For $z_o = r = 10^{-2}$ cm. $W_o = 1$. The build-up time $T_m = 0.14$ milliseconds and the maximum velocity is 50 cm./sec. The advance ΔW_m during the period is $0.8W_o$, i.e. $0.8r$ cm. These figures have reasonable orders of magnitude. One should bear in mind the limitations of the model (Equation 4), and the ideal physical conditions prescribed. The effect of whimsical factors such as the

deviation from the near-equilibrium conditions at the flow initiation stage, rugosity of the tube end, etc. cannot be estimated. Unless the interest of the experimenter is in the initial stages, it will be convenient always to have some liquid in the tube at time-zero as in the experiments of Siegel [9] . Imposing this condition circumvents the analytic difficulty of treating the initial development of the inter- face curvature and the attendant transient end-effects which occur when the tube end is first brought in contact with a liquid reservoir.

ACKNOWLEDGMENT

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APPENDIX

Brittin's equation is

$$(zz')' + b(zz') + \tfrac{1}{4}(z')^2 + cZ + d = 0 \tag{7}$$

with $-d$ the same physical quantity as P in equation (4a). The initial conditions are set in this manner: $\underline{i}$ "$z(0) = 0$", $\underline{ii}$ "The initial velocity V_o is assumed to be finite as it must be physically, and thus $\underline{is\ obtained\ from\ the\ differential}$ $\underline{equation}$ by setting $Z = 0$. $V_o = (-4/5d)^{\frac{1}{2}}$ ". (The italics and the numeration $\underline{i}$ and $\underline{ii}$ by the present writers).

We note: that (1) the 'feed-back' procedure in $\underline{ii}$ is not valid, since both conditions must be set independently; that (2) $Z_o = 0$ sets the initial $\underline{mass}$ and the initial kinetic energy equal to zero; and that (3) $Z_o = 0$ suggests the initiation of flow, while $Z_o' = (-4/5\ d)^{\frac{1}{2}}$, which is about 80 cm. per second in the representative system mentioned above, really applies to a later stage of the motion.

To illustrate this difficulty we proceed as follows: equation (5b), which is the general solution of (4b), includes all pairs of initial conditions and not only those applicable to spontaneous imbibition. Let us write $W = \varphi(T, W_o, W_o')$ and thus define a function φ of three variables. Now take W_o' arbitrary but bounded, and examine the behavior of $\partial\varphi/\partial T$ near $(0, 0, W_o')$. First let T go to zero, yielding:

$$\underset{W_o \to 0}{\text{Lim.}} \left. \frac{\partial \varphi}{\partial T} \right|_{T \to 0} = W_o'$$

as we wish i.e. $\partial\varphi/\partial T$ at $(0, 0, W_o') = W_o'$. Also $W_o'' = 1/W_o'$ which is Newton's law of motion in dimensionless units for unit force. Now let $W_o' \to 0$. This yields (there is a removable indeterminacy):

$$\underset{T \to 0}{\text{Lim.}} \left. \frac{\partial \varphi}{\partial T} \right|_{W_o' = 0} = 1$$

Thus the limit depends on the path. However, if $W_o = 0$ is excluded, the limit of $\partial \varphi / \partial \tau$ is W_o' independent of the path, and consequently $(0,0,W_o')$ is a singularity. The misbehavior at this point is immediately apparent for the integral of equation (7) near $(0,0,Z_o')$: the power series expansion about $t = 0$, contains terms with $1/Z_o$ as a factor.

FIGURES

Figure 1.   Initial stages of capillary rise.   Plot of inter-
face advance $\triangle W = W - W_o$, velocity W' and acceleration
W'' <u>vs</u> T.   Initial mass, 1 unit.

Figure 2.   Dynamics of initial impulsive transient as a function
of W_o, the effective initial mass (or length of
column).   Plot of maximum velocity attained W'
(max), T_m, time to attain W' (max), and $\triangle W_m$ the
interface advance <u>vs</u> W_o (log scale).

Figure 1
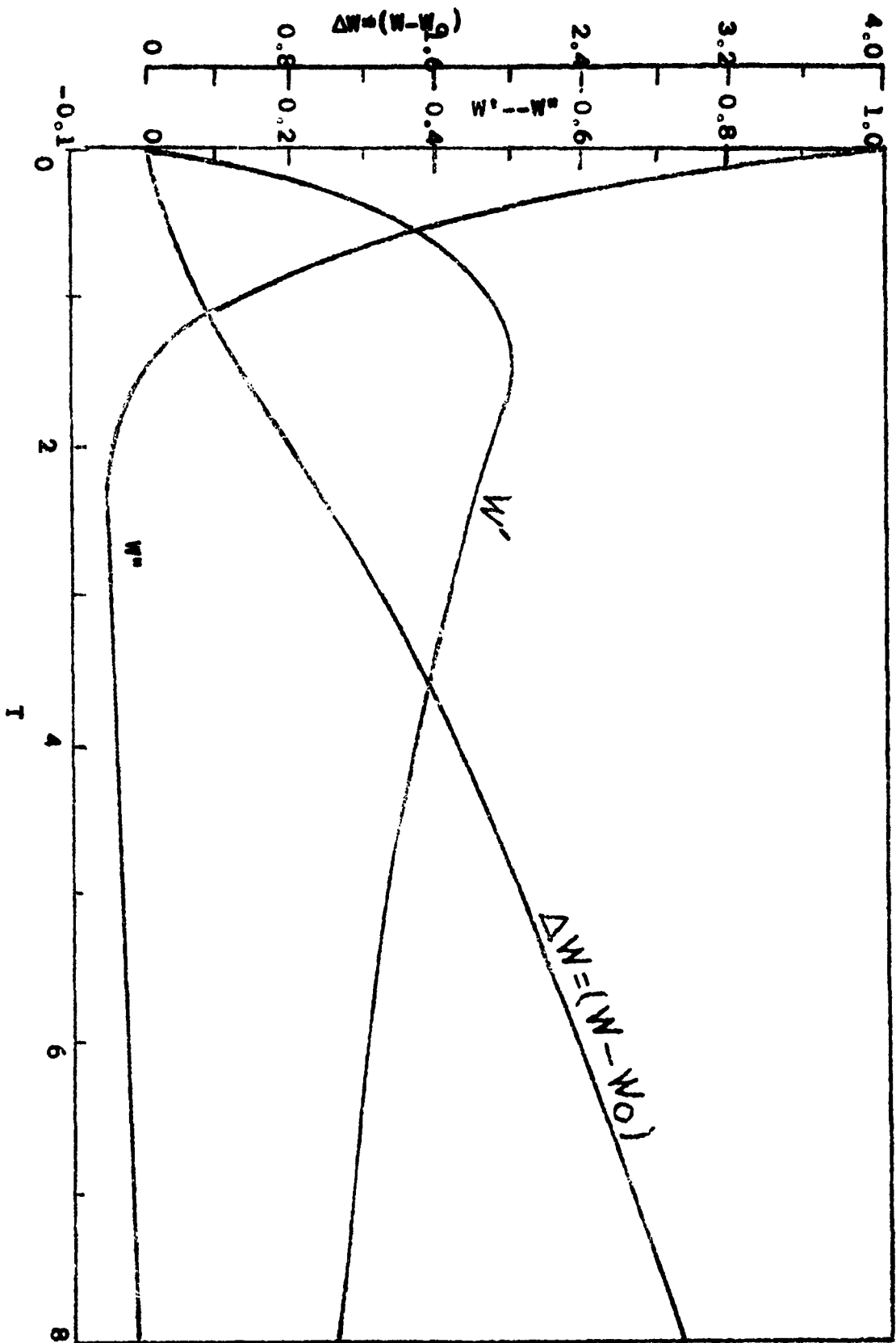
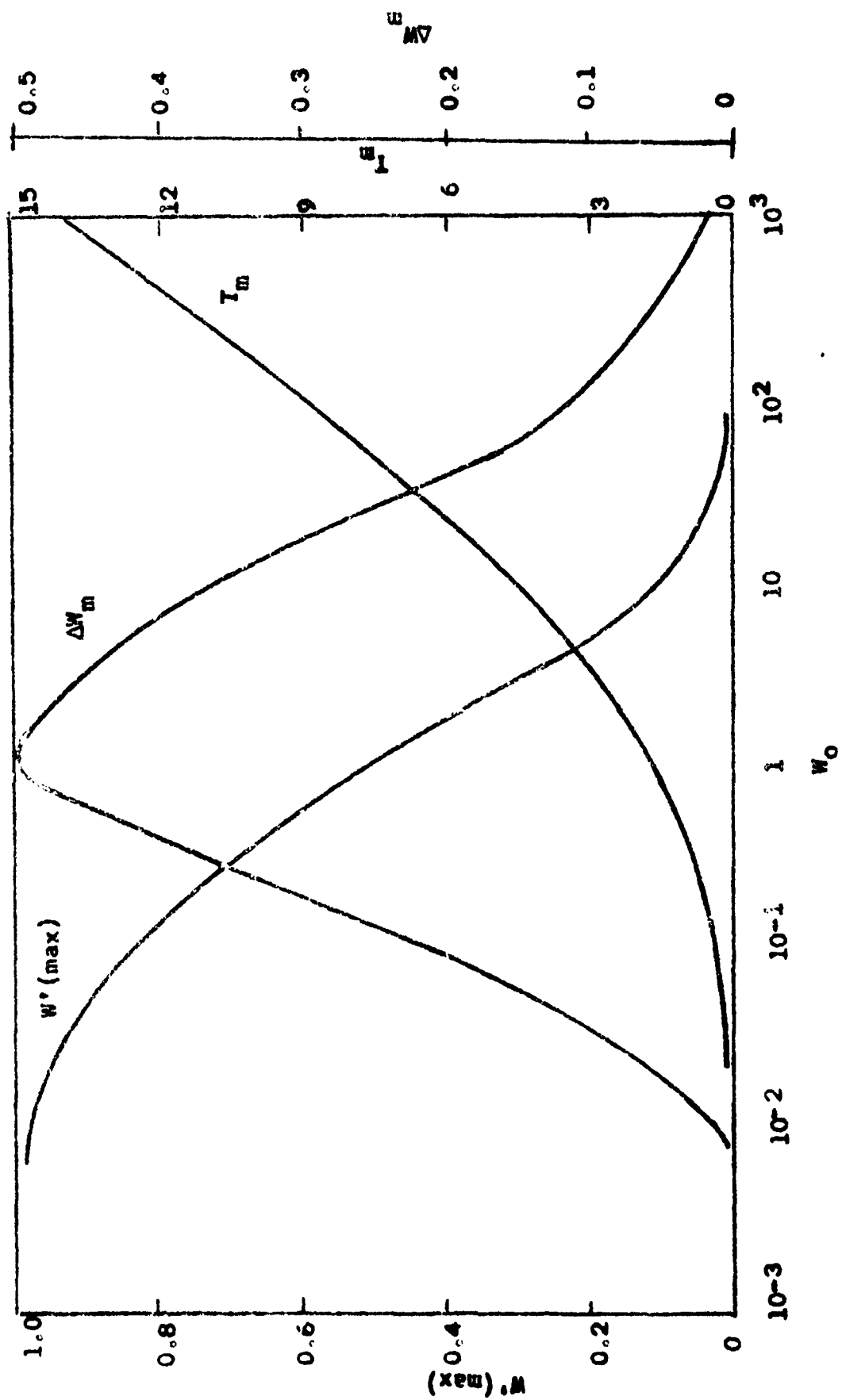Figure 2

Spectral Analysis of Structure of Porous Media
By
Harry Fara

Introduction

The geometrical quantities generally used to characterize a porous medium, such as porosity and specific surface, are over-all averages and are not sufficient to describe it exactly or identify it uniquely. It would be desirable to have criteria by which it can be decided by a simple analysis of photomicrographs whether two porous media are alike or not, and to what degree. Some preliminary ideas in this direction were outlined by Fara and Scheidegger[1].

Structure Function

It is obvious that in order to define "lower-order" structure factors (related (say) to porosity), recourse must still be made to statistics. We first construct a function which records not only the porosity but the "lower-order" density factors as well. An arbitrary line is assumed to be drawn through a given porous medium whose geometry is to be described. Points on the line are to be defined by giving their arc length S from an arbitrary chosen origin. Then, for certain values of S the line will pass through void spaces; for other values of S it will pass through filled spaces. We then introduce a function $f(S)$ of the arc length S defined as follows: the value of f is defined as $+1$ if the line at the point passes through a void space; it is defined as equal to -1 if the line at S passes through filled space. The function is a random "rectangular wave". (See f in figure 1).

We note the mean $f(S)$ is related to the porosity P by:

$$\bar{f} \simeq 2P-1 \tag{1}$$

i.e. in the limit as the interval over which f is defined is increased; the relationship (1) will tend toward equality. The definition of $f(S)$ can of course be modified so that the mean would yield P exactly but there is no advantage for the later analysis.

SPECTRAL ANALYSIS

The first term of a Fourier expansion of any function yields its mean value, so that (1) suggests a spectral decomposition of $f(S)$. We write formally

$$f(S) = \sum \left[A_n \cos \frac{2 \pi n S}{L} + B_n \sin \frac{2 \pi n S}{L} \right] \tag{2}$$

where L is the interval analysed and A_n, B_n are the Fourier co-efficients to be determined. They are given by:

$$A_n = \frac{2}{L} \int_0^L f(S) \cos \frac{2 \pi n S}{L} \, dS; \quad B_n = \frac{2}{L} \int_0^L f(S) \sin \frac{2 \pi n S}{L} \, dS \tag{3}$$

The sets of co-efficients (A_n, B_n) $n=0,1,2,3.....$ these characterize the porous medium. It is clear that any differences in substructure between two porous media will be reflected in differences in _some_ of the sets (A_n, B_n). For example, if the porosities P are identical then for $n=0$, we will have $A_0 = 2P-1$, and $B_0 = 0$ for both cases, but other sets will differ in general. The analysis can be simplified conceptually by the following considerations: the terms in (3) can be combined in pairs into terms containing a phase angle, which, however depends on the choice of origin, and thus is not relevant to our analysis. The new co-efficients C_n are given by:

$$c_n{}^2 = A_n{}^2 + B_n{}^2$$

The c_n then, are the true spectral densities. (It follows that $c_o = 2P-1.$).

Continuous Structure Function

The function $f(S)$ is only piece-wise continuous. Now it would be useful if the c_n reflected the substructure in as simple a way as possible, viz., by the most rapid convergence of the series (2), so that the c_n fall off as rapidly as possible with increasing n. The rate of convergence is higher for continuous functions. Rounding off the corners of $f(S)$ in some consistant fashion suggests itself. Each square wave in $f(S)$ was replaced by a sine wave of the same area and of the same half-wave length. The new function $F(S)$ is also shown in figure (1). The discontinuties are now in the first derivative. A simple analysis shows however, that the sine wave arcs introduce extraneous "noise".

The f and F of figure (1) are the structure functions of a sandstone of 26.3% porosity. The line is very short (0.17cm). The first 14 values of c_n for F and F were computed and are shown in table (1). The rectangular wave densities are plotted in figure 2.

Structure Profile

Figure 2 suggests how the results of the spectral analysis can be presented graphically to provide a structure "profile" of a porous medium. Of particular interest would be the peaks in the c_n, if any. It should be noted that the length used

in this particular case, (0.17cm) involving only 12 alternations
of void and filled regions is probably too small to make
figure 2 accurate, although the calculated porosity is
25.8% in comparison with the known value of 26.3%. One may
converge on the profile by analysing successive longer lengths
L_k. The spectral densities C_n must now be plotted vs. the
wave number $\nu = n/2L_k$ so that the various profiles can be
shown on one graph. The degree rate of convergence can be
noted at each stage.

Computation

Since the function $f(S)$ is either +1 or -1, computation
of the A_n, B_n, and C_n is relatively simple. Consider an
arbitrary "wave" of $f(S)$ with the discontinuities at a_o b_o,
a_1 b_1,...a_{n-1} b_{n-1},a_n. For the strips $(a_o b_o)$, $(a_1 b_1)$...
$(a_n b_n)$ set $f(S) = +1$; for the strips $(b_o a_1)$ $(b_1 a_2)$... $(b_{n-1} a_n)$
set $f(S) = -1$. The formulae (3) take on a simple form.

$$(L/2)_n = \sum_{k=0}^{n} \left[\int_{a_k}^{b_k} \cos \frac{2\pi n S}{L} \, dS \right] + \sum_{k=0}^{n-1} \left[\int_{a_k}^{b_k} \cos \frac{2\pi n S}{L} \, dS \right] \qquad (5)$$

with a similar formula for the B_n. Formula (5) can be further
reduced for numerical works since the intergrations can be per-
formed and the upper limits of the two integrals are the same.
The calculations can be programmed quickly for a digitial computer
for the general case; in each instance, a list of the a_k,b_k,
which can be read off a photomicrograph, and the length L_k
would be the only inputs necessary.

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TABLE 1.   Spectral Co-efficients of a
Porous Medium

n	Function f c_n	Function F c_n
0	0.464	0.465
1	0.267	0.268
2	0.606	0.714
3	0.272	0.457
4	0.154	0.052
5	0.277	0.289
6	0.232	0.498
7	0.244	0.474
8	0.439	0.478
9	0.260	0.247
10	0.353	0.410
11	0.142	0.110
12	0.158	0.154
13	0.229	0.278
14	0.088	0.085

Fig. 1

Fig. 2
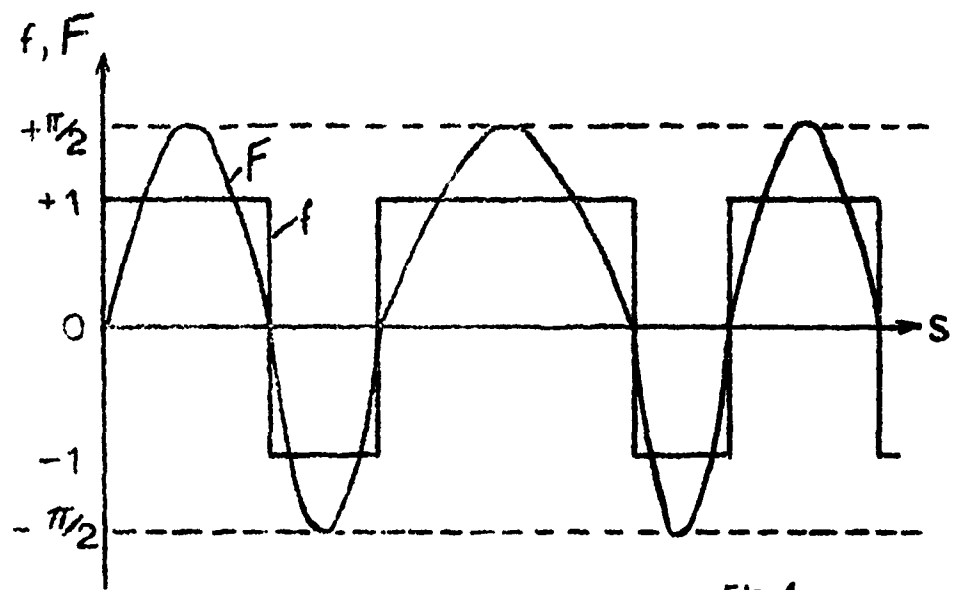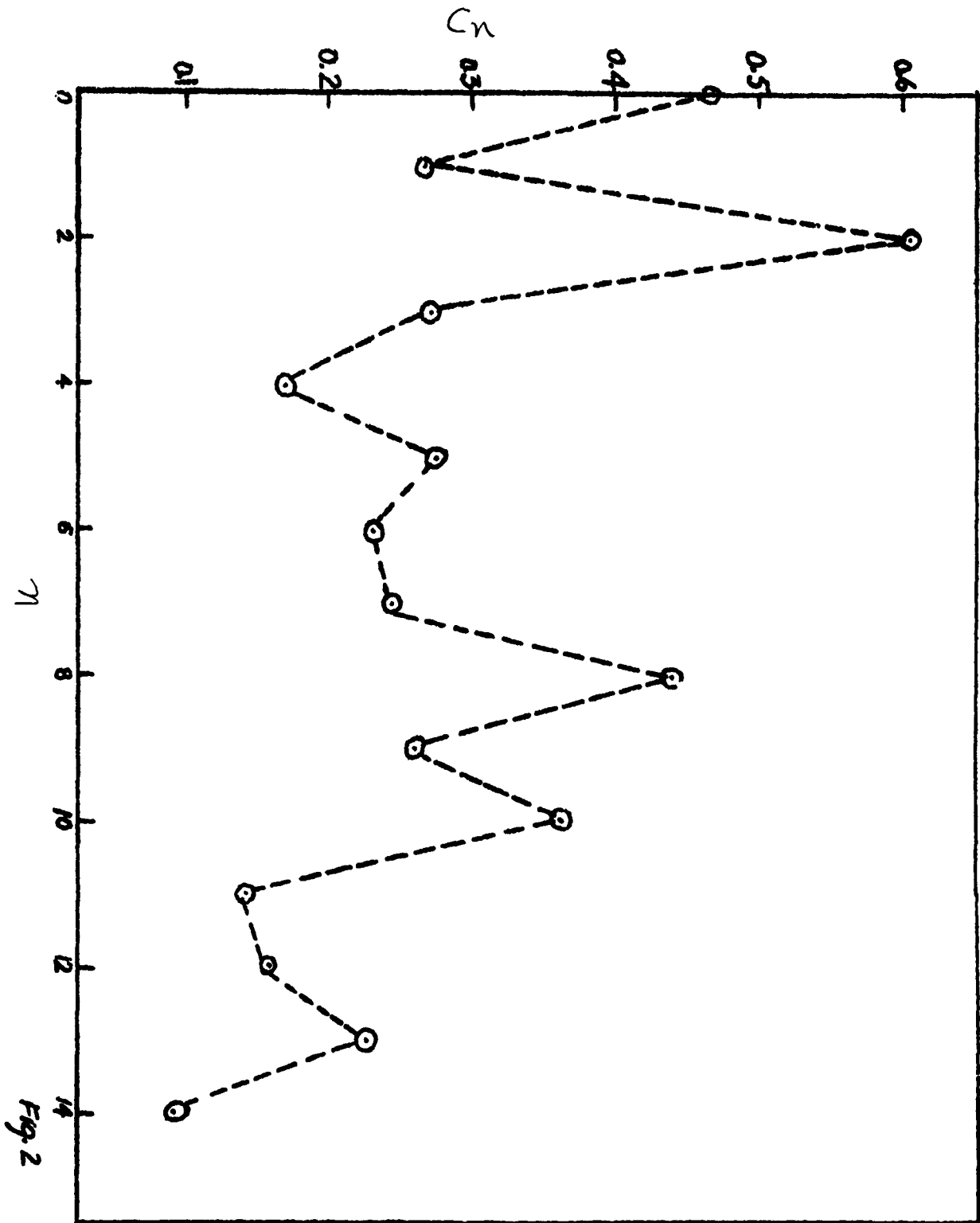
NUMERICAL INTEGRATION OF STOKES-NAVIER EQUATIONS
By
Harry Para, D. Geiger and W. Rose

The position is often taken* that the equations of
Stokes-Navier are perfectly general for the description of
all types of flow (e.g. turbulent as well as laminar) in
any space (e.g. porous media as well as unbounded space),
and under all conditions(unsteady as well as steady states).
The generality of this proposition (cf. Schlichting, Boundary
Layer Theory, McGraw-Hill, 1960) is its own limitation,
inasmuch as integrals of the resultant nonlinear, high-order
partial differential equations of motion are not known in
closed form even if well conditioned initial and boundary
condition could be specified (which, for the general case,
is not possible). On the other hand, approximations sometimes
are possible which lead to a fair description of the dynamics
of basically complex fluid motions (e.g. The Oseen approximation,
the Prandtl "boundary-layer" procedure, Stokes "creep flow",
etc.). We choose now to consider another possibility, namely
the "brute-force" numerical integration of the Stokes-Navier
equations by resort to finite-difference analogs as evaluated
by the   use of high-speed computer procedures.

As an elementary, albeit heuristically stimulating case
to consider, we choose the problem of Figure 1, where a set
of coupled pistons (separated by a distance, d move in steady
motion through a cylindrical tube of radius, a. Let us say
that the contained fluid is incompressible, and that the
velocity of piston movement is V_o. This case, which is mean-
ingful for an application to the fountain-effect problem

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*cf. Birkhoff, Hydrodynamics, A Study of Fact. Theory and
 Similitude, Second Edition, McGraw-Hill Book, 1960.

posed by Rose (Nature, July 15, 1961), is represented by:

$$\rho \, D\vec{v}/Dt = \mu \nabla^2 \vec{v} - \nabla P$$

and

$$\text{div} \, \vec{v} = 0 \tag{1}$$

which equations refer of course to a fixed Eulerian coordinate system. In Equation 1, $\vec{v}$ is the velocity vector having the axial component, u, and the radial component, w; ρ and μ are the fluid density and viscosity, respectively; and P is the pressure.

Taking these equations in cylindrical coordinates, (r, x, θ), and fixed to the coupled moving pistons; and further, assuming $\vec{v}$ independent of θ and time, we obtain:

$$S_R T_x - S_x T_R + (T/R) \, S_x = 2 \mu \, D^2 T \tag{2}$$

where

D^2 is the operator giving $D^2 S = RS_{RR} + S_{XX} = T$ (i.e. $D^2 T$ is the biharmonic of S).

$$S_x = r \, u$$

$$S_r = -r \, w$$

$$R = r^2/4$$

and where the subscripts, R, r and x, represent differentiation of the indicated function with respect to the subscript.

The no-slip boundary conditions for the case of viscous flow (cf. Figure 1) are:

$$S_x = 0 \text{ for } R = a^2/4 \text{ and for } x = \pm d/2$$

$$S_R = 0 \text{ for } x = \pm d/2$$

$$S_R = V_o \text{ for } R = a^2/4$$

With these boundary conditions imposed, the steady-state solution, $S = S(x,r)$, is to be found by a numerical iterative procedure with the finite-difference analog of Equation 2.

A preliminary study indicates that the iterations necessary to obtain convergence can be undertaken with economy on medium speed computers. Figure 2 depicts schematically the flow pattern which is suggested by physical intuition. To be noted is the fact that symmetries other than axial symmetry cannot be predicted a priori; moreover, it is likely that the singularities at the piston perimeters (arising because the velocity component, u, is not single-valued at these points) will introduce a difficulty.
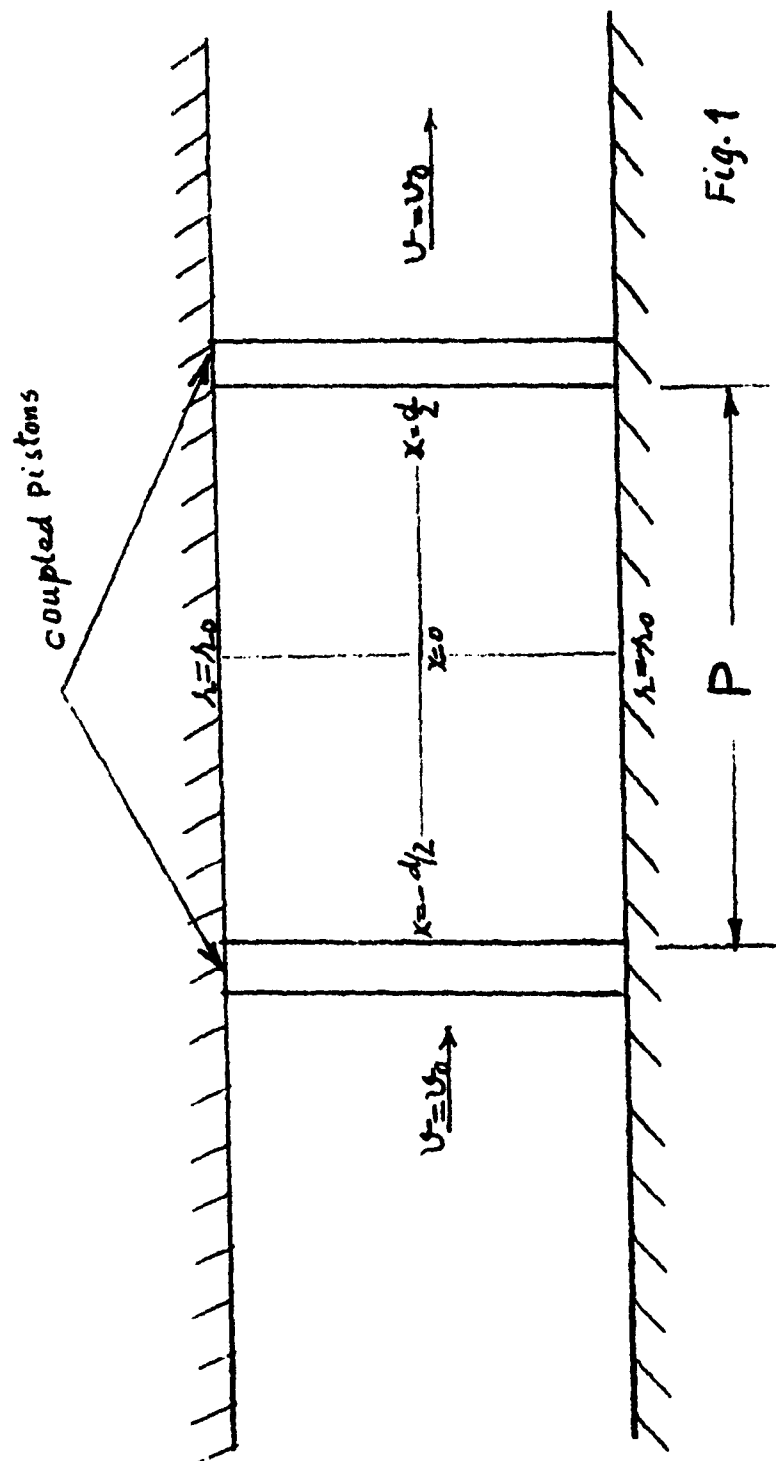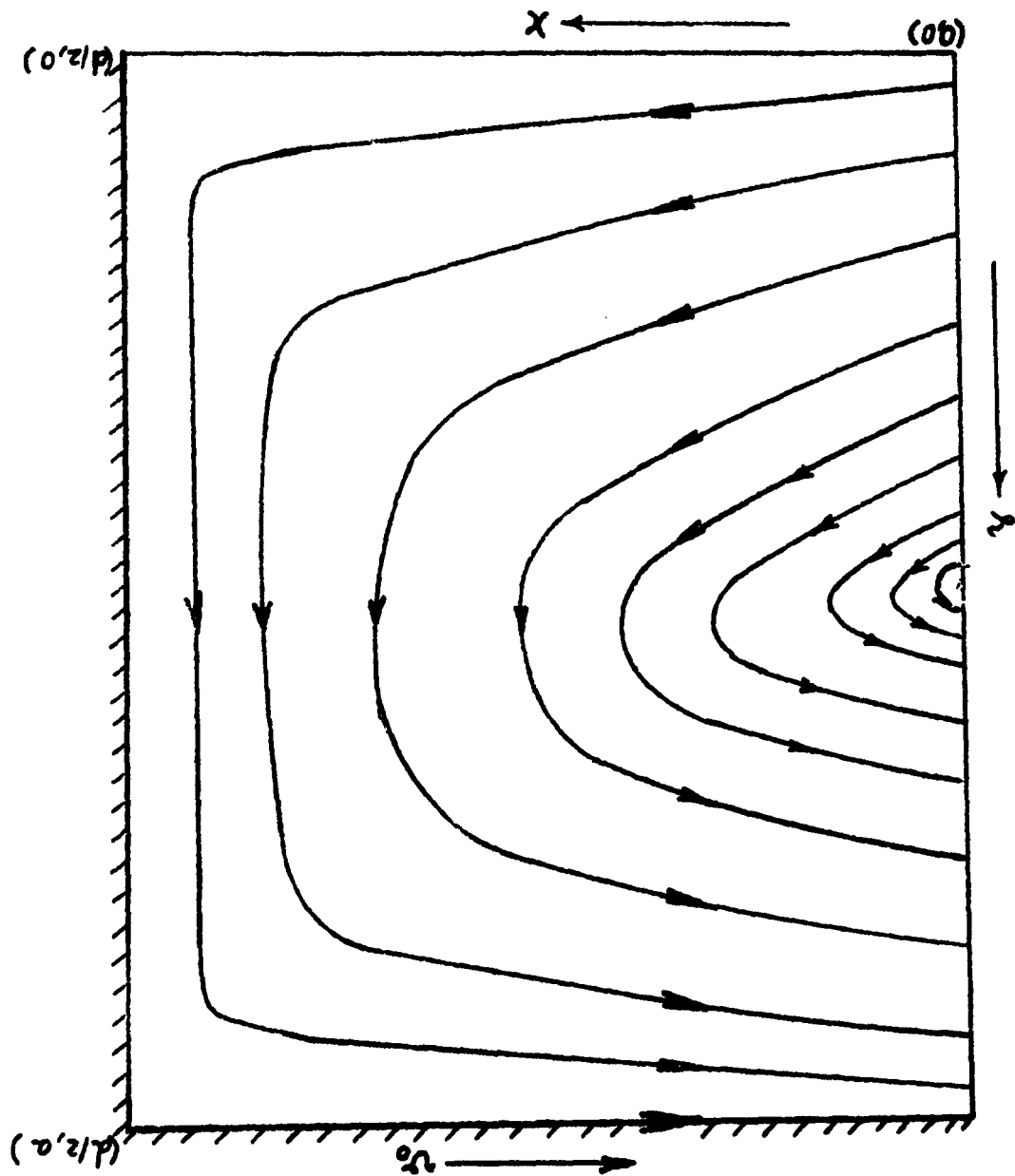
Fig.1

NETWORK ANALOGUES FOR ANALYSIS OF FLOW
THROUGH POROUS MEDIA

III Note on a "Discriminating" Routine For A Step-Wise
Desaturation Process

By
Harry Fara

The applicability of electric network analogues to the
study of flow through porous media is limited by the large
number of components required to sensibly represent an actual
sample of porous material. This is true of course for physical
models constructed with resisters, etc.[1] but applies
as well to "networks" which are to be solved by pencil-and-
paper operations. We are concerned with the latter case,
for which this limitation can be overcome in part by the use
of high-speed digital computers. The flow chart of a programme
for solving a steady-state single phase problem, written for
the University of Illinois computer "Illiac" was described in
section I. In section II (Seventh Quarterly Progress Report)
an analysis was presented of a procedure suitable for a
step-wise desaturation process i.e. when a wetting fluid B
replaces a liquid A.

A critical operation in the   procedure, whether
effected manually or with a computer, is keeping track of
possible paths for fluid motion. At the ith stage (say) the
"pressure" imposed is such that (1) pores of sizes $\geq S_i$ are
now possible flow channels for the invading fluid (see II);
but for B to replace A in a given pore we must also have (2)
an exit path to the sink boundary for A and (3) a continuous
path of pores of sizes $\geq S_i$ from the source boundary to the
given pore. If these three conditions hold, then a pore is

allowed to "pop" i.e. fluid B is considered to have re-
placed fluid A. Application of these criteria is simple
but tedious in manual work[2,3]. For machine processing a
sequence of disciminating binary operations resulting in
yes-no decisions is required which will duplicate the results
of visual discriminations.

Consider Figure 1, a miniature rectangular network con-
sisting of resistors (conductances) represented by dashed
and solid lines and linked at the nodes, (the circles). The
solid lines represent pores now containing fluid B, the dis-
placing phase, while the dashed lines represent pores still
containing fluid A. The source boundary is at $x=0$. The
conductances to the lift of $x=0$ (solid lines) which are not
part of the network conveniently fix the line as the source
of fluid B. The desaturation process is at some
intermediate stage (say the ith) with the network partitioned
as shown. The conductivities of each can now be calculated
(see I).

We now assume that the pressure is increased so that
pores of size $\geq S_i$ are now possible channels for B; these may
include some of sizes $=S_{i-1}$ which did not "pop" at earlier
stages because conditions (2) and/or (3) were not satisfied.
The scanning procedure begins at this point. It is clear
that we may not proceed in any arbitrary way. It is the
logical (machine-programmed) attack of this problem which is
to form our future work.

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SOURCE, FLUID A

X_o

Fig.1. Miniature Network

X_s

SINK FOR FLUIDS A,B

(ε)
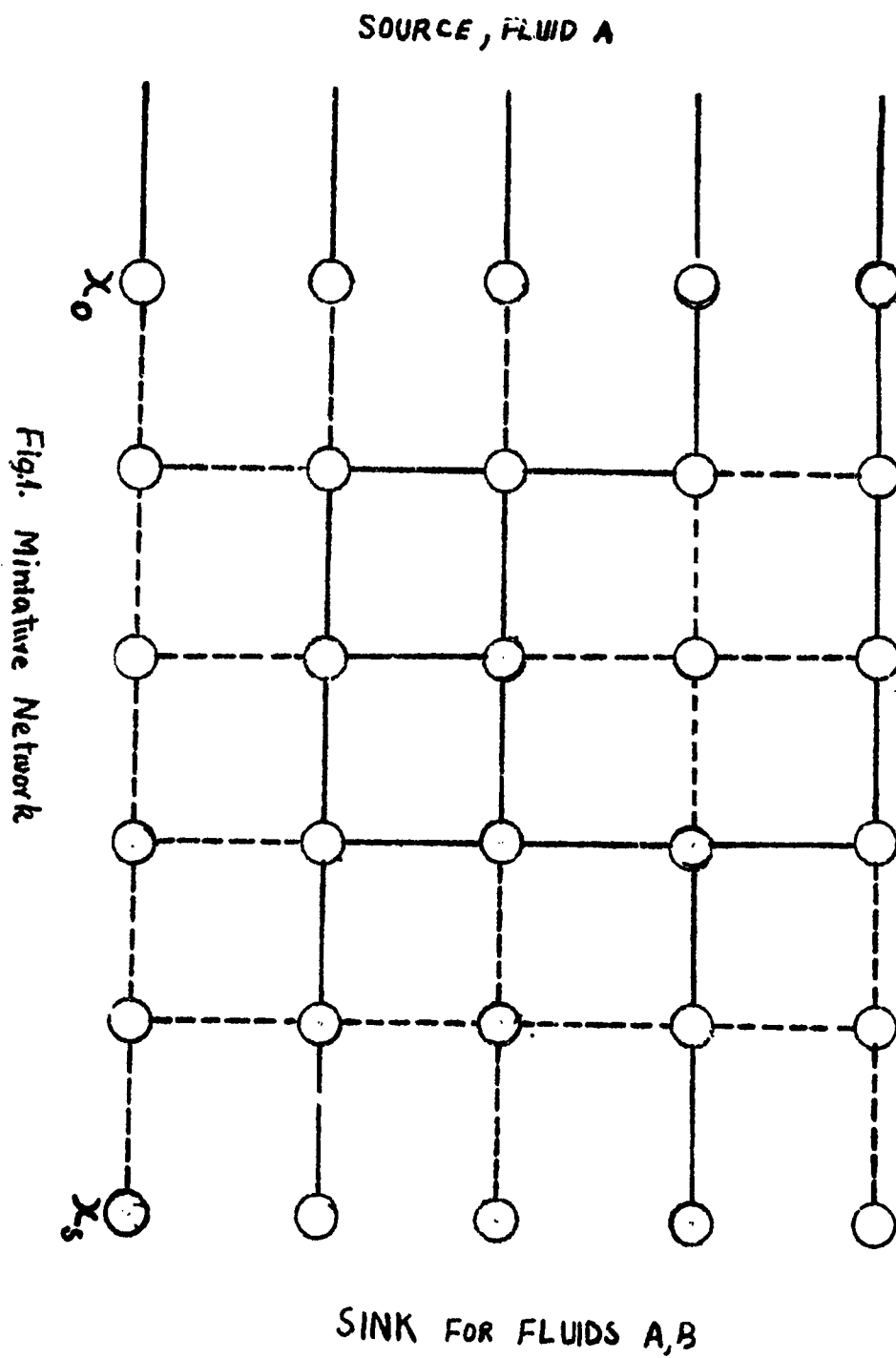
Contact Angle Acceleration Dependency
By
N. Chaudhari and Walter Rose

As a continuation of the work described by the writer
in the sixth and seventh Quarterly Progress Reports of this
series, experiments were undertaken where the decay of a
manometer was observed. In such a system, accelerations-
decelerations occur, so that the observation of the inter-
face position of the manometric fluid as a function of time
provides the data against which hypothesis can be tested
about the dependency of the contact angle of moving fluid-fluid
interfaces on the velocity, and on the time-derivative of the
velocity of interface movement.

In the experiments reported here, data were obtained
using manometers where the diameter of the tube arms either
was 0.185 cm., 0.200 cm., or 0.200 cm. Dow Corning (Number 200)
silicon oils were used as the manometric fluid, having
viscosities either of 10, 7, 5 or 3 centipoises. The whole
system was kept under temperature control at 25°C. The position
of the trailing manometer interface, Z, as a function of time,
was recorded with a Pathe (Webo) 16 mm. movie camera.

Results

In all cases, the experimental data could be represented
by a linear relationship between the logarithm of the ratio
(Z_M/Z_M-Z) and time, where Z_M marks the time-infinity position
of manometer balance. According to theory, deviations from
linearity only are to be expected during the first milli-
seconds of the impulsive transient which follows the initiation
of motion. Theory also predicts that the slope of the straight

line function will have a high value if the advancing
and receding contact angles in the leading and trailing
manometer arms have the same value through the decay; while
a sensibly lower value for the slope is predicted on the
assumption that the receding contact angle has some fixed
(time-invariant) value (say, zero), and on the assumption
that the advancing contact angle has a time-dependency repre-
sented by:

$$\cos \theta_A = 1 + M_1 Z' + M_2 Z'' \tag{1}$$

where θ_A is the advancing contact angle of the leading in-
terface, and Z' and Z'' are the velocity and the acceleration
(deceleration) of the motion, and M_1 and M_2 are constants.

Equation 1 is arrived at by postulating that θ_A is
both velocity and acceleration-dependent, as given by taking
the first two terms of a double-Taylor series expansion.
This form was suggested by taking note of the observations
of Rose and Heins (cf. previous progress reports of this
series, cf. also, J. Colloid Science, December, 1961) that
the cosine of θ_A is linearly related to velocity for steady-
state motions.

In any case, a measurement of the slope of the experi-
mental data plot of $\ln(Z_M/Z_M-Z)$ versus time, according to
theory (i.e. a closed analytic solution of the manometer
problem is known)*, provides a way to evaluate the constants
of Equation 1. Table I shows results obtained for the 3
centistoke oil, where the values of M_1 and M_2 have been obtained
by a least-squares fit of the data making use of an IBM 650
program. Here it is seen that the magnitude of the acceleration-
dependency (M_2) is of the order of only 10-percent of the

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*cf. MS thesis of N. Chaudhari, University of Illinois,
 February, 1962.

velocity dependency (M_1), and is of opposite sign. The
meaning of this is that the velocity-dependency (as measured
by negative values of M_1) serves to slow down the manometer
decay in accordance with the principle of LeChatelier as
predicted by Rose and Heins (loc.cit.); while the acceleration-
dependency acts similarly because in the present instance of
manometer decay, the acceleration is negative (i.e. the decel-
eration is positive) after the first impulsive instants when
the motion is initiated.

Table I also suggests that both velocity and acceleration-
dependency increase as the tube size increases. This represents
a trend which is predicted by the argument that the capillary
forces are less effective in large tubes (than small tubes)
to maintain high curvature of the moving interface, therefore,
flattening by the hydrodynamic forces (i.e. the "fountain effect"
of Rore, Nature, July 15, 1961) occurs.

Since Table I indicates that the acceleration-dependency
is small compared to the velocity-dependency, the neglect
of the higher order and degree terms of the Taylor expansion
of Equation 1 apparantly is justified, similarly, there appears
to be no reason to consider expansions which take $\cos \theta_\Lambda$
as a function of $(Z', Z'', Z''', \ldots)$. Indeed, Siegel (J. Applied
Mechanics, June 1961) has even dropped the M_2 term of Equation
1 in his analysis of capillary rise (i.e. a decelerating system),
therefore, we show in Table II our calculations of M_1
(i.e. the velocity-dependency) which represent the best
(least-squares) fit of our data on the assumption that M_2 can
be set equal to zero. Here again, a general increase in M_1
is observed correlating with an increase in tube size; moreover,
M_1 also appears to increase with increasing viscosity. Both

of these trends are consistant with theoretical expectations
as implied above (cf. MS thesis of N. Chaudhari, loc.cit.).

TABLE I

Tube Size	$0.11C_m$ R.	$0.1C_m$ R.	$0.0915 C_m$ R.
M_1	0.156	0.0926	0.084
M_2	0.0122	0.0094	0.00036

TABLE II

Tube Size	$0.11C_m$ R.	$0.1C_m$ R.	$0.0915 C_m$ R.
Viscosity of oil			
10	0.75	0.94	-
7	0.5	0.377	0.201
5	0.65	0.63	0.438
3	0.28	0.083	0.053